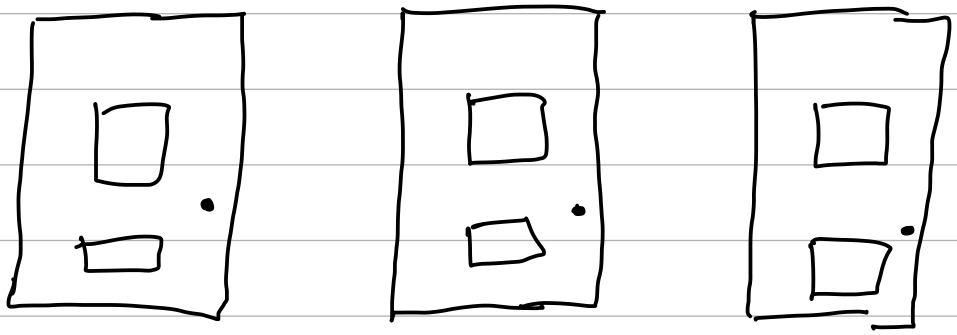
Lecture 8

Monty Hall Problem



- One door has a car
- The rest have donkeys
- You pick a door
  - ↳ The guest opens a door that has a donkey
    - ↳ the guest asks if you want to change your pick.

- You pick door 1 originally, host opens door 2

- $P(C_1) = \frac{1}{3}$ • $S = C_1 \cup C_1^c$

- $C_1 \rightarrow$ "car is behind door 1"

- $P(C_3) = \underbrace{P(C_3 \mid C_1)}_{0} P(C_1) + \underbrace{P(C_3 \mid C_1^c)}_{1} \underbrace{P(C_1^c)}_{1-\frac{1}{3}}$

- $P(C_3) = \frac{2}{3}$
  - ↳ So you should change your vote.

Bayes Formula

- Let (S, P) be a probability space, and let $B_1, \dots, B_n, \dots \subset S$ be events that form a partition of S, that is:

$$B_i \cap B_j = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad \bigcup_{n \geq 1} B_n = S$$

with $P(B_n) > 0$ for all $n \geq 1$. Then for any event $A \subset S$ with $P(A) > 0$, and $k \geq 1$, the conditional probability of B_k given A is

$$P(B_k \mid A) = \frac{P(A \mid B_k) P(B_k)}{P(A)} = \frac{P(A \mid B_k) P(B_k)}{\sum_{n \geq 1} P(A \mid B_n) P(B_n)}$$

<u>Proof</u>:

$$P(B_k \mid A) = \frac{P(B_k \cap A)}{P(A)} = \frac{P(A \mid B_k) P(B_k)}{P(A)}$$

$$P(A) = P(A \mid B_1) P(B_1) + \dots + P(A \mid B_n) P(B_n)$$

<u>Cor</u>:

$$P(B \mid A) = \frac{P(A \mid B) P(B)}{P(A)} = \frac{P(A \mid B) P(B)}{P(A \mid B) P(B) + P(A \mid B^c) P(B^c)}$$

$\boxed{\text{Ex}}$ Factory has 2 machines that produce light bulbs.

↳ Machine A produces 60 % of bulbs with defect rate of 2 %.

↳ Machine B produces 40 % of bulbs with defect rate of 5 %.

a) What is probability that randomly selected bulb is defective.

"defective"

$$P(D) = P(D \mid A)P(A) + P(D \mid B)P(B)$$

$$= (0.02)(0.6) + (0.05)(0.4)$$

$$= 0.032$$

b) If a bulb is found to be defective, what is the probability that it was produced by machine B?

$$P(B \mid D) = \frac{P(B \cap D)}{P(D)} = \frac{P(D \mid B)\,P(B)}{P(D)} = \frac{(0.05)\cdot(0.4)}{(0.032)}$$

$\boxed{\text{Ex}}$ Smallpox vs Chickenpox

<u>Given</u>:

$P(\text{spots} \mid \text{smallpox}) = 0.9$

$P(\text{spots} \mid \text{chickenpox}) = 0.8$

<u>Want to know</u>:

$P(\text{smallpox} \mid \text{spots})$

$P(\text{chickenpox} \mid \text{spots})$

a) <u>Assume</u> $P(\text{smallpox}) = P(\text{chickenpox})$

$$P(\text{smallpox} \mid \text{spots}) = \frac{P(\text{spots} \mid \text{smallpox}) \cdot P(\text{smallpox})}{P(\text{spots})} = \frac{0.9\,(P)}{P(\text{spots})}$$

$$P(\text{chickenpox} \mid \text{spots}) = \frac{P(\text{spots} \mid \text{chickenpox})\,P(\text{chickenpox})}{P(\text{spots})} = \frac{0.8\,(P)}{P(\text{spots})}$$

b) Assume $P(\text{smallpox}) = 0.01$, $P(\text{chickenpox}) = 0.1$

Then:

$$P(\text{smallpox} \mid \text{spots}) = \frac{(0.9)(0.01)}{P(\text{spots})}$$

$$P(\text{chickenpox} \mid \text{spots}) = \frac{(0.8)(0.1)}{P(\text{spots})}$$

- This teaches us its important to have good assumptions.

$\boxed{\text{Ex}}$ Polygraph Test

- L = "the person lies"
- L_p = "the polygraph says the person lies"
- T = "person said the truth"
- T_p = "polygraph says person said truth"

- Given:

$$P(L_p \mid L) = 0.95 = P(L_p^c \mid L^c) = P(T_p \mid T)$$

- So $P(L_p \mid T) = 1 - 0.95 = 0.05$

- So $P(L \mid L_p) = \dfrac{P(L_p \mid L)\,P(L)}{P(L_p \mid L)P(L) + P(L_p \mid L^c)\,P(L^c)} = \dfrac{0.95\,(P)}{0.95(P) + 0.05 + (1-P)} = \dfrac{1}{2}$

$\boxed{Ex}$ A disease affects 1% of a population. A medical test correctly identifies an infected individual with probability 0.98, and incorrectly reports infection for healthy individual with probability 0.05.

a) $P(\underset{\text{person tests positive}}{+}) = P(+ \mid \underset{\text{infected}}{I})P(I) + P(+ \mid \underset{\text{Healthy}}{H})P(H)$ $\qquad S = I \cup H$

$0.98 \cdot 0.01 + 0.05 \cdot 0.99$

b) $P(I \mid +)$

- use Bayes formula

$$P(I \mid +) = \frac{P(+ \mid I)\, P(I)}{P(+)}$$

$\boxed{Ex}$ A package shipped using one of 3 delivery services:

$P(S_1) = 0.4$, $P(S_2) = 0.35$, $P(S_3) = 0.25$

The probability that a package arrives late are:

$P(L \mid S_1) = 0.01$, $P(L \mid S_2) = 0.03$, $P(L \mid S_3) = 0.06$

a) probability random package arrives late?

↳ Law of total probability

↳ $P(L) = P(L \mid S_1)P(S_1) + P(L \mid S_2)P(S_2) + P(L \mid S_3)P(S_3)$

↳ because $S = S_1 \cup S_2 \cup S_3$

b) Given package was late, what is probability that it was shipped using service S_3?

↳ use Bayes

$$P(S_3 \mid L) = \frac{P(L \mid S_3)P(S_3)}{P(L)}$$ ← found in part a

$\boxed{\text{Ex}}$ An urn contains three types of coins:

- 40% are fair coins
- 40% are biased coins that land heads with probability 0.6
- 20% are biased coins that land heads with probability 0.9

- Given coin landed heads, what is probability it was fair coin?

↳ use Bayes!

$$P(F \mid H) = \frac{P(H \mid F)P(F)}{P(H)} = \frac{P(H \mid F)P(F)}{P(H \mid F)P(F) + P(H \mid B_1)P(B_1) + P(H \mid B_2)P(B_2)}$$

fair coin

heads

first biased coin

second biased coin

$\boxed{\text{Ex}}$ A student answers MCQ right with probability 0.85 if they studied and 0.25 if they didn't study. Suppose 70% of students studied.

a) What is prob that random student answers question correctly?

↳ Law of total probability because $S = \text{study} \cup (\text{study})^c$

$$P(\text{correct}) = P(\text{correct} \mid \text{study})P(\text{study}) + P(\text{correct} \mid (\text{study})^c)\, P[(\text{study})^c]$$