Lecture 7

Rmk: In Applications:

- A_+ is a future event
- A_0 is a present event
- A_- is a past event

Ex Alex flips fair coin n+1 times while Sean flips the same coin n times. Alex wins if the number of heads she gets is strictly greater than the number of heads Sean gets. What is probability that Alex wins?

$P(A)$: ?

After n steps: A gets X heads
S gets Y heads

↳ only 3 possibilities
$X > Y$, $X = Y$, $X < Y$

$$P(A) = P(A \cap \{X>Y\}) + P(A \cap \{X=Y\}) + \underbrace{P(A \cap \{X<Y\})}$$

0, since A can only match the # of heads S got.

$$P(X>Y) = P(Y>X) = P$$
$$P(X=Y) = 1 - 2P$$

$$P(A \cap \{X > Y\}) = \overbrace{P(A \mid \{X > Y\})}^{1} \, \overbrace{P(\{X > Y\})}^{p}$$

$$P(A \cap \{X = Y\}) = \underbrace{P(A \mid \{X = Y\})}_{\frac{1}{2}} \, \underbrace{P(\{X = Y\})}_{1-2p}$$

$$P(A) = 1 \cdot p + \tfrac{1}{2}(1-2p) + 0 = \tfrac{1}{2}$$

<u>Law of Total Probability (Thm)</u>: Let (S, P) be a probability space, and let $B_1, \dots, B_n \subset S$ be events that form a partision of S; that is,

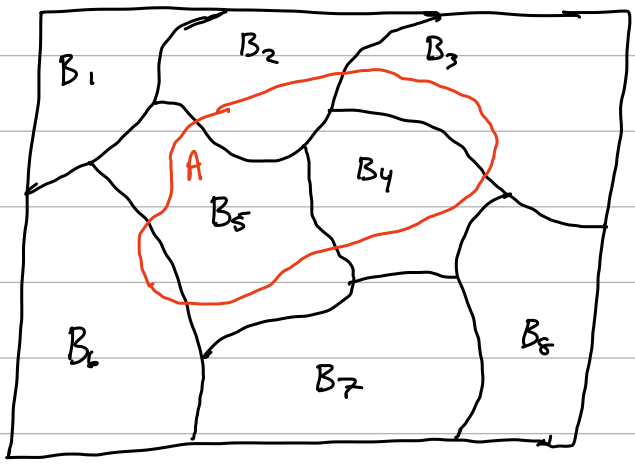
$$B_i \cap B_j = \emptyset \text{ for } i \neq j \quad \text{and} \quad \bigcup_{i=1}^{n} B_i = S$$

with $P(B_i) > 0$ for all i. Then for any event $A \subset S$,

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

<u>Rmk</u>: The Law of Total Probability allows us to compute probabilities by conditioning on cases that exhaust all possibilities.

- We have $P(A) = P(A \mid B_1)P(B_1) + \dots + P(A \mid B_n)P(B_n)$



$$P(A) = P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3) + P(A \mid B_4)P(B_4) + P(A \mid B_5)P(B_5) + P(A \mid B_6)P(B_6)$$

$\boxed{\text{Ex}}$ On a given day Probability of snow is 0.3 and probability that it doesn't snow is 0.7. If it snows, Terry arrives late with probability 0.4. If it does not snow, Terry arrives late with probability 0.1. What is probability that Terry arrives late to class?

L = Terry is late S = snow S^c = no snow

$$P(L) = \underbrace{P(L|S)}_{0.4}\underbrace{P(S)}_{0.3} + \underbrace{P(L|S^c)}_{0.1}\underbrace{P(S^c)}_{0.7}$$

$$0.4 \cdot 0.3 + 0.1 \cdot 0.7 = 0.19$$

$\boxed{\text{Ex}}$ An urn contains b black balls and r red balls. A ball is chosen at random and discarded. Without knowing its color, a second ball is drawn. What is the probability that the 2nd ball drawn is black?

B_k = k^{th} ball is black, $P(B_2) = ?$

$$P(B_2) = \underbrace{P(B_2|B_1)}_{\frac{b-1}{b+r-1}}\underbrace{P(B_1)}_{\frac{b}{b+r}} + \underbrace{P(B_2|B_1^c)}_{\frac{b}{b+r-1}}\underbrace{P(B_1^c)}_{\frac{r}{b+r}}$$ because $S = B_1 \cup B_1^c$

$$\frac{b-1}{b+r-1} \cdot \frac{b}{b+r} + \frac{b}{b+r-1} \cdot \frac{r}{b+r} = \frac{b}{b+r}$$

notice this is the same as drawing the 1st ball to be black

$\boxed{\text{Ex}}$ Consider a set of n markers, each with a corresponding cap. A child removes all caps and then randomely assigns the caps back to the markers, one by one.

a) What is probability that no marker recieves its original cap?

N = "no markers has its own cap"

$$P(N) = 1 - P(N^c)$$

M = "at least 1 gets its own cap"

$$N^c = M$$

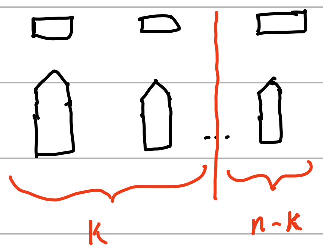
M_k = "k-th marker gets its own cap"

$$M = M_1 \cup M_2 \cup \dots \cup M_n$$ ← Not a union of disjoint events, So use inclusion exclusion principle

$$P(M) = P(M_1 \cup M_2 \cup \dots \cup M_n) = P(M_1) + \dots + P(M_n)$$
$$- P(M_1 \cap M_2) - P(M_1 \cap M_3) - \dots$$
$$+ P(M_1 \cap M_2 \cap M_3) + \dots$$
$$- \dots$$

$$= \binom{n}{1} P(M_1) - \binom{n}{2} P(M_1 \cap M_2) + \binom{n}{3} P(M_1 \cap M_2 \cap M_3) - \dots$$

$$P(M_1 \cap M_2 \cap \dots \cap M_k) = \frac{(n-k)!}{n!}$$



$$P(M) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!}$$

$$P(M) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}$$ Same

$$P(N) = 1 - P(M) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

as $n \to \infty$ $P(N) = \frac{1}{e}$

b) Whats the probability of the k^{th} marker getting its own cap?

$P(M_k) = ??$

$N_k =$ "non of first k-1 markers gets k^{th} cap"

$N_k \cup N_k^c = S$

$$P(M_k) = P(M_k | N_k)P(N_k) + \underbrace{P(M_k | N_k^c)}_{0}P(N_k^c)$$

because cap was already used

$$P(M_k | N_k) = \frac{1}{[n-(k-1)]}$$

because we don't touch the k^{th} cap

$$P(N_k) = \frac{(n-1)\cdot((n-1)-1)\cdot \ldots \cdot ((n-1)-(k-2))}{n\cdot(n-1)\cdot\ldots\cdot(n-(k-2))} = \frac{n-k+1}{n}$$

So $P(M_k) = \frac{1}{n}$