Lecture 6

LAST TIME: The conditional probability of A given B.

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

Def: Given a sample space S, and $A \subset S$. A collection of events $A_1, A_2, \dots A_n$ is a partition of A if

i) $A_i \cap A_j = \emptyset$ if $i \neq j$

ii) $A = \bigcup_{n\geq 1} A_n$

Proposition: Given a probability space (S, P), and an event $B \subset S$ such that $P(B) \neq 0$.

Then $A \longmapsto P(A|B)$ defines a probability function on S. i.e. it satisfies 3 properties:

(i) $0 \leq P(A|B) \leq 1$, $\forall A$

(ii) if $B \subset A \subset S$ then $P(A|B) = 1$, in particular $P(S|B) = 1$

(iii) if $(A_n)_{n\geq 1}$ is a partition of A then $P(A) = \sum_{n\geq 1} P(A_n)$

Cor: $P(A^c|B) = 1 - P(A|B) \quad \forall A \subset S$

$P(A_1 \cup A_2 | B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2 | B)$

Independence

Def: Given a probability space (S, P) and two events $A, B \subset S$ are called independent if

$P(A \cap B) = P(A) \cdot P(B)$

Notation for independence: $A \perp\!\!\!\perp B$ or $A \perp B$

Rmk: if $P(A) = 0$ or $P(B) = 0$

$(A \cap B) \subset A$, $P(A \cap B) \leq P(A)$

$\rightarrow P(A \cap B) = P(A)P(B) \rightarrow A \perp\!\!\!\perp B$

Rmk! if $P(A)P(B) \neq 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$A \perp\!\!\!\perp B$ i.f.f. $P(A|B) = P(A)$

Prop: $A \perp\!\!\!\perp B \to A \perp\!\!\!\perp B^c$

Proof:
$$\begin{aligned} P(A \cap B^c) &= P(A \setminus (A \cap B)) \\ &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B^c) \\ &\to A \perp\!\!\!\perp B^c \end{aligned}$$

$\boxed{\text{Ex}}$ Toss a coin twice. $S = \{HH, HT, TH, TT\}$

$A = \{\text{the } 1^{st} \text{ is } H\} = \{HH, HT\}$

$B = \{\text{the } 2^{nd} \text{ is } H\} = \{TH, HH\}$

$C = \{\text{they are different}\} = \{TH, HT\}$

$P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$

$P(A \cap B) = P(\{HH\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)$

$P(B \cap C) = P(\{TH\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(B)P(C)$

$A \perp\!\!\!\perp B$, $B \perp\!\!\!\perp C$, $A \perp\!\!\!\perp C$ so they are pairwise independent

$P(\underbrace{A \cap B \cap C}_{= \emptyset}) = 0 \neq P(A)P(B \cap C)$

Def: (a) The sequence of events $A_1, A_2, \dots, A_n$ is called independent if for any $k \geq 1$ and any $i_1 < i_2 < \dots < i_k$ we have:

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \dots \cdot P(A_{i_k})$$

(b) The sequence of collection of events $C_1, C_2, \dots, C_n$ is independent if any sequence of events $A_1, A_2, \dots, A_n$ s.t. $A_k \in C_k$ is independent.

Rmk: Sequence of events $\{A_1, A_2, \dots\}$ is independent $\longleftrightarrow$ the sequence of collections $\{A_1, A_1^c\}$, $\{A_2, A_2^c\}, \dots, \{A_n, A_n^c\}$

$\boxed{\text{Ex}}$ Roll a dice until the first 6 appears

$B_n = \{$ we get our first 6 at the n^{th} roll $\}$

$A_k = \{$ the k^{th} roll is 6 $\}$

$$B_n = A_1^c \cap A_2^c \cap \dots \cap A_{n-1}^c \cap A_n$$

- The independence assumption $\rightarrow P(B_n) = P(A_1^c) \cdot P(A_2^c) \cdot \dots \cdot P(A_{n-1}^c) \cdot P(A_n) = \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)$

$\boxed{\text{Ex}}$ Roll a die 10 times. What is the probability to get 6 exactly 2 times?

$B_{ij} = \{$ got 6 at exactly i^{th} and j^{th} rolls $\}$

$P(B_{ij}) = \left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$

$B_{12} = A_1 \cap A_2 \cap A_3^c \cap \dots \cap A_{10}^c$

$$P(\text{exactly 2 "6"}) = P(\cup B_{ij}) = \sum_{i<j} P(B_{ij})$$
$$= \binom{10}{2} P(B_{ij})$$

Def: Given a probability space (S, P) and $A, B, C, \subset S$ s.t. $P(C) \neq 0$.

- We say that A, B are conditionally independent given C if:

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

Prop: (Markov Property)

- Given a probability space (S, P). $A_-, A_0, A_+ \subset S$ such that $P(A_0) > 0$. Then A_+, A_- are conditionally independent given A_0 i.f.f. $P(A_+ \mid A_- \cap A_0) = P(A_+ \mid A_0)$

Proof: $P(A_+ \mid A_- \cap A_0) = \dfrac{P(A_+ \cap (A_- \cap A_0))}{P(A_- \cap A_0)}$

$$= \frac{P(A_+ \cap A_- \mid A_0)\, P(A_0)}{P(A_- \mid A_0)\, P(A_0)}$$

$$= \frac{P(A_+ \cap A_- \mid A_0)}{P(A_- \mid A_0)}$$

$$= P(A_+ \mid A_- \cap A_0) = P(A_+ \mid A_0)$$