Lecture 5

- $\boxed{\text{Ex}}$ An urn contains 10 red balls, 10 green balls, and 10 blue balls. You draw 5 balls at random without replacement. What is the probability that you do NOT get all the colors?

R = "No red balls"

G = "No green balls"

B = "No blue balls"

$P(R \cup B \cup G) = ?$

$$= P(R) + P(B) + P(G)$$

$$- P(R \cap B) - P(R \cap G) - P(G \cap B)$$

$$+ \underbrace{P(R \cap G \cap B)}_{=0}$$

$$= 3\left[P(R) - P(R \cap B)\right]$$

$$P(R) = \frac{\binom{20}{5}}{\binom{30}{5}}, \qquad P(R \cap B) = \frac{\binom{10}{5}}{\binom{30}{5}}$$

So $3\left[P(R) - P(R \cap B)\right] = P(R \cup G \cup B) = 3\left[\frac{\binom{20}{5} - \binom{10}{5}}{\binom{30}{5}}\right]$

• $\boxed{\text{Ex}}$ In how many ways can we write $k \in \mathbb{N}$ as a sum of n integer non-negative terms $k_1, \dots, k_n$? What if $k_1, \dots, k_n$ all positive?

$k_1 + k_2 + \dots + k_n = k \quad k_i, k \geq 0$

$3 + 2 + 0 + 1 = 6 \quad n=4, k=6$

$\bullet\bullet\bullet | \bullet\bullet || \bullet$

??? ??????

• So we have 9 spots to put the fences.

• So its $\binom{9}{3}$ ways to put fences

• So for general problem above its $\binom{k+n-1}{n-1}$

$\boxed{\text{Ex}}$ Now what if we have the same question but all terms are <u>positive</u> (no zeros)

$2+2+1+1 = 6 \quad \begin{matrix} n=4 \\ k=6 \end{matrix}$ • Now fences <u>cannot</u> be next to each other

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• So now you have $(k-1)$ slots to choose from so general solution is $\binom{k-1}{n-1}$

- Conditional Probability

$\boxed{\text{Ex}}$ Somebody rolls a dice

A = "the outcome is a double"

B = " the sum of the numbers is 10"

What is the probability of A given B has occured?

$$S = \{(n_1, n_2)\}_{n_1, n_2 \in \{1,2,\cdots,6\}}$$

$$B = \{(6,4), (4,6), (5,5)\}$$

$$A = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$$

$$P(A|B) = \frac{1}{3}$$

$$P(B|A) = \frac{1}{6}$$

Def: Let (S, P) be a probability space. If $A, B \subset S$ are events with $P(B) \neq 0$, then the conditional probability of A given B is the real number $P(A|B)$ defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Rmk: The definition above implies $P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$

$\boxed{Ex}$ Alice and Bob are playing a gambling game. Each rolls a fair six-sided die, and the player with the higher number wins. If they tie, they roll again. If Alice has just won, what is the probability that she rolled a 5.

Conceptual Approach

$A =$ "Alice Won"

$R_i =$ "Alice rolled i"

- We are looking for $P(R_5 \mid A)$

↳ Assume alice rolled a, and Bob rolled b. (a,b)

$A = \{(2,1), (3,2), (3,1), (4,3), (4,2), (4,1), (5,4), (5,3), (5,2), (5,1)$
$(6,5), (6,4), (6,3), (6,2), (6,1)\}$

$|A| = 15$

The # of outcomes in A where Alice rolled 5 is 4.

So $P(R_5 \mid A) = \frac{4}{15}$

$\boxed{Ex}$ A family has 2 pets (only pets are dog and cat)

i) What is probability that both pets are dogs given one pet is a dog.

$D = \text{dog}$

$C = \text{cat}$

So $P(\{DD\} \mid (DD \cup DC \cup CD)) = \dfrac{P(DD)}{P(DD \cup DC \cup CD)} = \dfrac{(1/4)}{(3/4)} = \dfrac{1}{3}$

ii) What is probability both pets are dogs given one of the pets is a dog born on Tuesday.

$D = \text{Dog}$ $\quad D_T = $ "Tuesday Dog"

$C = \text{Cat}$ $\quad D_* = $ "Non-Tuesday Dog"

so $P\left(\{DD\} \mid (D_T D \cup D_* D_T \cup C D_T \cup D_T C)\right)$

$= \dfrac{P(D_T D \cup D_* D_T)}{P(D_T D \cup D_* D_T \cup C D_T \cup D_T C)}$

NOTE: $D = D_T \cup D_*$, and $D_T \cap D_* = \emptyset$

$D_T D = D_T D_* \cup D_T D_T$

• Since the intersection is $\emptyset$

$P(D_T D \cup D_* D_T) = P(D_T D) + P(D_* D_T)$

$P(D_T D) = \underbrace{\frac{1}{14}}_{\frac{1}{2} \cdot \frac{1}{7}} \cdot \frac{1}{2}$ $\qquad P(D_* D_T) = \underbrace{\frac{6}{14}}_{\frac{1}{2} \cdot \frac{6}{7}} \cdot \underbrace{\frac{1}{14}}_{\frac{1}{2} \cdot \frac{1}{7}}$

• So $P(D_T D \cup D_* D_T) = \left(\frac{1}{14}\right)\left(\frac{1}{2}\right) + \left(\frac{6}{14}\right)\left(\frac{1}{14}\right)$

$P(D_T D \cup D_* D_T \cup C D_T \cup D_T C) = P(D_T D) + P(D_* D_T) + P(C D_T) + P(D_T C)$

$\frac{1}{14}\left(\frac{1}{2}\right) + \left(\frac{6}{14}\right)\left(\frac{1}{14}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{14}\right) + \left(\frac{1}{14}\right)\left(\frac{1}{2}\right)$

• So $\dfrac{P(D_T D \cup D_* D_T)}{P(D_T D \cup D_* D_T \cup C D_T \cup D_T C)} = \dfrac{13}{27}$

• Notice $\left(\dfrac{P(D_T D \cup D_* D_T)}{P(D_T D \cup D_* D_T \cup C D_T \cup D_T C)}\right) > \dfrac{P(DD)}{P(DD \cup DC \cup CD)}$