Lecture 4

Lecture 4 Problem Set (page 2)

1a) $\binom{7}{3}$

1b) $(7)_3$

2a) A

2b) B

3a) True, since $\frac{(n)_k}{k!} = \binom{n}{k}$ so $\binom{n}{k} k! = (n)_k$

3b) False, $\binom{n}{k}$ only represents the # of ways to choose k people out of n.

4) $(52)_3$, since the order here matters

5) $\binom{5}{2} = \frac{5 \cdot 4}{2!}$, $(5)_2 = 5 \cdot 4$

Colorings of n objects with m colors.

- The number of colorings of n objects into m colors such that n_1 objects are colored with color 1, n_2 objects with color 2, ..., n_m objects with color m where $n_1 + n_2 + \ldots + n_m = n$ is

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdots \binom{n_{m-1}+n_m}{n_{m-1}} \binom{n_m}{n_m}$$

$$\frac{n!}{n_1!\,\cancel{(n-n_1)!}} \cdot \frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}} \cdot \frac{\cancel{(n-n_1-n_2)!}}{n_3!\,(n-n_1-n_2-n_3)!} \cdots \frac{\cancel{(n-n_1-\ldots-n_{m-2})!}}{(n_{m-1})!\,\cancel{n_m!}} \cdot \frac{\cancel{n_m!}}{n_m!} = \frac{n!}{(n_1)!\,(n_2)!\ldots(n_m)!}$$

$\boxed{\text{Ex}}$ How many colorings of 15 distinct balls are there, such that 5 of them are blue, 3 are green, and 7 are red?

Ans: $\binom{n}{n_1, n_2, n_3} = \frac{n!}{n_1! n_2! n_3!}$ $\boxed{\binom{15}{5,3,7} = \frac{15!}{5!\,3!\,7!}}$

Newtons Binomial Formula

- Let n be a positive integer. The Newton binomial Formula is the identity:

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n} y^n$$

"Proof": $(x+y)(x+y)(x+y) = x^3 + yxx + xyx + xxy + yyx + yxy + xyy + y^3$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Notice coefficients correspond to $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$

$$\underbrace{(x+y)\dots(x+y)}_{n} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

- Corrolaries:
  - Let $x = y = 1$
    ↳ you get $(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$
  - Let $x = 1, y = -1$
    ↳ you get $(1-1)^n = 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \dots$
  - Now if you sum the two results you get:
    $2^n = 2\binom{n}{0} + 2\binom{n}{2} + 2\binom{n}{4} + \dots$
  - Also:
    $2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots$

Problem Set Poker Hands Problem

- you are dealt a 5-card poker hand from a 52-card deck, without replacement

a) what is probability that the hand contains exacty K hearts?

↳ total 5-card hands $= \binom{52}{5}$

↳ "good" events : $\binom{13}{K}\binom{39}{5-K}$

choosing K cards that are hearts

choosing the other cards such that they are not hearts

So the answer is
$$\frac{\binom{13}{K}\binom{39}{5-K}}{\binom{52}{5}}$$

b) What is the most likely number of hearts in the hand?

- If you compute, you will see $K=1$ is the answer.

c) What is probability that the hand contains two pairs, but is not a full house?

- A pair means the number or letter matches
- A full house is where you have 2 pairs, and the 5^{th} card is one of the 2 numbers/letters already in your hand. $\boxed{\text{Ex}}$: 7♢, 7♡, 7♠, 5♣, 5♠

choosing our 3 types

choosing which types give us a pair

$$P(\text{2 pairs, not full house}) = \frac{\binom{13}{3}\binom{3}{2}\underbrace{\binom{4}{2}\binom{4}{2}}_{\text{choosing suits for the pairs}}\binom{4}{1}}{\binom{52}{5}}$$

choosing a suite for last card