Lecture 3

Ex A standard deck of 52 playing cards is given:

ⓐ How many two-card hands consist of one heart and one club?

ⓑ How many two-card hands consist of one "number" card that is a heart and one "picture" card that is a club?

$$\underbrace{2,3,4,5,6,7,8,9,10}_{\text{number}},\ \underbrace{J,Q,K,A}_{\text{picture}}$$

a) 13 options for hearts
13 options for club
so $13 \times 13 = 13^2$

b) 9 options for "number card" that is a heart
4 options for "picture card" that is a club
so $9 \times 4 = 36$

Sampling with Replacement

- An urn contains n balls labeled 1 through n. Sampling with replacement is the experiment consisting of the following steps:
    - extract one ball at random from the urn
    - record the label of the extracted ball
    - return the extracted ball to the urn

- After k ordered samplings, an outcome is $(l_1, l_2, \dots, l_k)$ $1 \le l_i \le n$
- The number of possible outcomes is n^k.

<u>Def</u>: Let m be a nonnegative integer, We set

$$m! = \begin{cases} 1 & \text{if } m=0 \\ 1\cdot 2\cdot 3\cdot \ldots \cdot m & \text{if } m>0 \end{cases}$$

- The number <u>$m!$</u> is called <u>m factorial</u>.

- Let us also introduce the notation

  $(x)_k = x(x-1)\ldots(x-k+1)$ where k is an integer, and $\forall x \in \mathbb{R}$

  ↳ This is called the <u>pochhamer symbol</u>.

  Pochhammer symbol

$\boxed{Ex}$: $(20)_3 = 20\cdot(20-1)\cdot(20-2)$

<u>Sampling without replacement</u>

- An urn contains n balls labeled 1 through n. Sampling <u>with</u> replacement is the experiment consisting of the following steps:
  - extract one ball at random from the urn
  - record the label of the extracted ball
  - discard the extracted ball

- The number of possible outcomes:

$$A_{k,n} = n(n-1)\cdot\ldots\cdot(n-k+1) = (n)_k = \frac{n!}{(n-k)!}$$

where n is the number of items and you are sampling k items.

1st Problem in the Problem set

- urn contains balls labeled 0 to 99
- 5 balls drawn <u>without</u> replacement
- winner must guess all five numbers in the order they were drawn
- What are odds of winning the lottery?

$$P(\text{win}) = \frac{1}{\#\text{ of possible outcomes}}$$

$$\#\text{ of possible outcomes}: 100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 = (100)_5$$

Problem #2

- n people in a room
- each person birthday is independently and uniformly distributed over 365 days
- What is probability 2 people share the same birthday?

$\#$ of possible outcomes : 365^n

A_n = "at least 2 people have the same birthday"

$P(A_n) = 1 - P(A_n^c)$

A_n^c = "all n people have different bday"

$$P(A_n^c) = \frac{365 \cdot 364 \cdot \ldots \cdot (365-n+1)}{365^n} = \frac{(365)_n}{365^n}$$

$$P(A_n) = 1 - \frac{(365)_n}{(365)^n}$$

Permutations

Def: A permutation of objects labeled $1, 2, \ldots, k$ is a bijection:

$$\ell : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$$

$\boxed{\text{Ex}}$ Permutations for $\{1,2\}$: $(1,2), (2,1)$

- The number of permutations of k objects is $k!$

Rmk: Stirling's formula: $k! \approx \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$ as $k \to \infty$

$\boxed{\text{Problem \#3}}$

- How many 5 card hands can be formed from a standard 52-card deck?

number of 5 card hands $= 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ ← but here the order matters

for example lets say we get: card 1, card 2, card 3, card 4, card 5

↳ any permutation of these is the same hand

↳ so we need to divide by the # of permutations.

$$\text{ans} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

Combinations

$\binom{n}{k}$ or C_n^k (read n choose k) is the number of combinations of k objects out of n.

$$C_n^k = \binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$$

- $\binom{n}{k}$ also called binomial coefficients

Pascal Triangle:

$$\begin{array}{lccccccccccc}
\binom{0}{0} & & & & & & 1 & & & & & \\
\binom{1}{k} & & & & & 1 & & 1 & & & & \\
\binom{2}{k} & & & & \underset{\binom{2}{0}}{1} & & \underset{\binom{2}{1}}{2} & & \underset{\binom{2}{2}}{1} & & & \\
\binom{3}{k} & & & 1 & & 3 & & 3 & & 1 & & \\
\binom{4}{k} & & 1 & & 4 & & 6 & & 4 & & 1 & \\
\binom{5}{k} & 1 & & \underset{1+4}{5} & & \underset{4+6}{10} & & 10 & & 5 & & 1
\end{array}$$

Properties of Pascal Triangle:

1. $\binom{n}{k} = \binom{n}{n-k}$
2. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
3. $\binom{n}{0} + \binom{n}{1} + \dots \binom{n}{n} = 2^n$ ← this is the number of different colorings if there are n objects to color and 2 available colors.