Lecture 27

• The Law of Large Numbers

↳ $n \in \mathbb{N}$. Let X_n be bernoulli random variable that takes value 1 if we get a one at the n^{th} roll and 0 otherwise.

↳ So $X_n \sim Ber(\frac{1}{6})$ $\forall n$

$\mu = \mathbb{E}[X_n] = \frac{1}{6}$

$X_1 + X_2 + \ldots + X_n$ describes the # of 1's we get after the first n rolls of the die.

$\overline{X}_n = \frac{X_1 + X_2 + \ldots + X_n}{6}$ Sample mean or emperical mean represents the fraction of the first n rolls that result in 1.

Thm: The Law of Large Numbers

- Suppose that $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with mean μ and finite variance σ^2. Denote by $\overline{X}_n$ the emperical mean $\overline{X}_n = \frac{1}{n}(X_1 + \ldots + X_n)$

- Then for any $\varepsilon > 0$ we have

$$P(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$$

- In particular $\lim_{n\to\infty} P(|\overline{X}_n - \mu| > \varepsilon) = 0$, $\forall \varepsilon > 0$

Proof !

Note that $\mathbb{E}[X_1 + \dots + X_n] = \sum_{i=1}^{n} \mathbb{E}[X] = n\mu \to \mathbb{E}[\overline{X}_n] = \mu$

- Moreover, X_i are independent $\to var[X_1 + \dots + X_n] = n\sigma^2 \to var\left[\frac{X_1 + \dots + X_n}{n}\right]$

$$= var[\overline{X}_n] = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

Hence the stdev of $\overline{X}_n$ is $\frac{\sigma}{\sqrt{n}}$

- Fix $\varepsilon > 0$, Chebyshev's inequality $\to P(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{var[\overline{X}_n]}{\varepsilon} = \frac{\sigma^2}{\varepsilon n}$

$$\frac{\sigma^2}{\varepsilon^n} \xrightarrow[n\to\infty]{} 0 \qquad \blacksquare$$

- Def : Let (X_n) be a sequence of random variables and μ a constant. We say that X_n Converges in probability to μ, and we write

$$X_n \xrightarrow{\mathbb{P}} \mu$$

if for any $\varepsilon > 0$ $\lim_{n\to\infty} \mathbb{P}(|X_n - \mu| > \varepsilon) = 0$

Rmks:

1. Finiteness of the variance is not needed for LLN.
2. Strong Law of Large Numbers (SLLN): The emperical mean $\overline{X}_n$ converges almost surely to μ.
3. "$X_n \to \mu$ as $n \to \infty$" $= \bigcap_{\varepsilon > 0} \bigcup_{N \geq 1} \bigcap_{n \geq N} \left\{ |\overline{X}_n - \mu| < \varepsilon \right\}$
4. "Strong" law because almost sure convergence implies convergence in probability.