Lecture 26

Moment Generating Function (mgf)

- A function that contains all the info about your random variable.

- $X \sim$ random variable (discrete or continuous)
  ↳ It is s-integrable for any $s \geq 1$: $\int_{-\infty}^{\infty} x^s \cdot p(x)dx < \infty$

  $\sum_{k=1}^{\infty} x_k^s \, p(X = x_k) < \infty$

- We define moments of X: $\mu_1(X) = \mathbb{E}[X]$

  $\mu_2(X) = \mathbb{E}[X^2]$, ..., $\mu_k = \mathbb{E}[X^k]$

- **Def**: Mgf of X, $\mu_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu_k(X)$

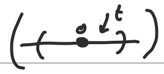
- **Prop**: $\mu_X(t) = \mathbb{E}[e^{t(X)}]$ (mgf)

- Proof: $\mu_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}[X^k] = \sum_{k=0}^{\infty} \mathbb{E}\left[\frac{t^k}{k!} X^k\right] = \mathbb{E}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k\right)$

  $= \mathbb{E}\left(\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right)$, Taylor series for e^x: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

  $= \mathbb{E}\left(e^{tX}\right)$

Thm: Suppose X and Y are 2 random variables and suppose that

$\forall t$ in some open interval around 0 

$$M_X(t) = M_Y(t)$$

THEN: $X \sim Y$ (identically distributed) $\iff$ $\mathbb{P}(X \le c) = \mathbb{P}(Y \le c)$ $\forall c$

Prop:

a) Let X be a random variable with a MGF $M_X(t)$ then $M_K(X) = M_X^{(k)}(0)$

b) If $X_1, X_2, \ldots, X_n$ are independent random variables, then

$$M_{X_1 + \ldots + X_n} = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$$

Ex Bernoulli

$$M_X(t) = \mathbb{E}(e^{tX}) = e^{t \cdot 0}(1-p) + e^t p = (1-p) + pe^t$$

Ex Binomial (p, n)

Sum of n independent Bernoulli variables

By prop (b) $M_X(t) = [(1-p) + pe^t]^n$

Ex Geometric (p)

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k=1}^{\infty} e^{tk} \underbrace{P(X=k)}_{(1-p)^{k-1}p} = p \sum_{k=1}^{\infty} e^{tk}(1-p)^{k-1} = pe^t \sum_{k=1}^{\infty} (e^t \cdot (1-p))^{k-1}$$

$$= \frac{pe^t}{1 - e^t(1-p)}$$

Ex $\text{Exp}(\lambda)$: pdf: $p(x) = \lambda e^{-\lambda x}$, $x \ge 0$

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(\lambda - t)x} dx$$

$$= \frac{\lambda}{\lambda - t}$$

Prop: Let $X_1, X_2, \ldots X_n$ be independent r.v's that are identically distributed, $X_i \sim Exp(\lambda)$

Then $N = X_1 + X_2 + \ldots + X_n \sim Gamma(n, \lambda)$

Proof: $M_N(t) = M_{X_1}(t) \cdot \ldots \cdot M_{X_n}(t) = \left[M_{X_1}(t) \right]^n = \left(\frac{\lambda}{\lambda - t} \right)^n =$ $M(t)$ for gamma (r, λ)

Poisson Process

- Stream of events that occur at times $s_1 < s_2 < s_3 < \ldots$
- Define $T_1 = S_1$, $T_2 = S_2 - S_1$, $T_3 = S_3 - S_2, \ldots, T_n = S_n - S_{n-1}, \ldots$

- Suppose $T_1, T_2, \ldots, T_n, \ldots$ are identically distributed $Exp(\lambda)$

- Define $N(t) = \#$ of buses that will arrive during $[0, t]$ time intervals.

- Question: What is the pmf of $N(t)$

  - Start with $\mathbb{P}[N(t) = 0] = \mathbb{P}(T_1 > t) = e^{-\lambda t}$

  - Next: $\mathbb{P}[N(t) = n] = \mathbb{P}(S_n \le t < S_{n+1}) = \mathbb{P}(S_n \le t) - \mathbb{P}(S_{n+1} \le t)$

    $F_n(t)$ $\quad F_{n+1}(t)$

- Observe: $S_2 = S_1 + (S_2 - S_1) = T_1 + T_2$

  $S_3 = S_1 + (S_2 - S_1) + (S_3 - S_2) = T_1 + T_2 + T_3$

  $\vdots$

  $S_n = T_1 + T_2 + \ldots + T_n \sim Gamma(n, \lambda)$

bc prof made mistake above

↓

- $P[N(t)=n] = F_{n+1}(t) + F_n(t)$

$$= \underbrace{\frac{\lambda^{n+1}}{\Gamma(n+1)}}_{n!} \int_0^t s^{n+1} e^{-\lambda s}\, ds - \frac{\lambda^n}{\underbrace{\Gamma(n)}_{(n-1)!}} \int_0^t s^n e^{-\lambda s}\, ds$$

$$F_{n+1}(t) - F_n(t) = \frac{-\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s}$$

So $P[N(t)=n] = \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s} \sim Poi(t\lambda)$