Lecture 25

• $\boxed{Ex}$ Suppose that the # of customers (1 hour) at lotter shop has poisson distribution with mean 5. Each customer buys a # of lottery tickets, indep. of other customers, and this number has a Poisson distribution with mean 2.

$$N \sim Poi(5)$$
$$T_i \sim Poi(2)$$

The total # of tickets is $T = T_1 + T_2 + \dots + T_N$

$$G_T(s) = G_N[G_{T_i}(s)] = e^{5(e^{2(s-1)}-1)}$$

$$G_{Poi(\lambda)}(s) = e^{\lambda(s-1)}$$

Generic form : $G_X(s) = P(X=0) + P(X=1)s^1 + P(X=2)s^2 + \dots$

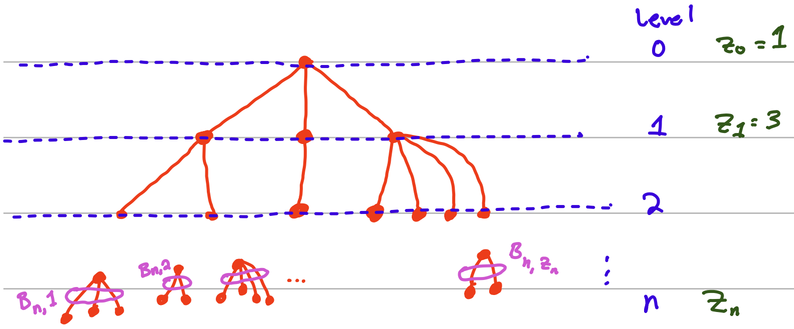
$$\mathbb{P}(T=0) = G_T(0) = e^{5(e^{-2}-1)}$$

$$\mathbb{E}[T] = \mathbb{E}[N]\,\mathbb{E}[T_i]$$

$N \sim Poi(5)$
$T_i \sim Poi(2)$ so $\mathbb{E}[T] = (5)(2)$ since in Poisson distribution, both the expectation and variance is given by the parameter.

$$var[T] = 2\cdot 5 + 2^2\cdot 5 = 30$$

Ex Branching Process



Assumptions

(i) $Z_0 = 1$

(ii) The family sizes of each bacteria are independent rand. var.

(iii) The family sizes of each bacteria have the same pmf as a fixed r.v. B with values in $\{0, 1, 2, \dots\}$

pmf of B : p_B

pgf : $G_B(s)$

$G_B(s) = p_B(0) + p_B(1)s + p_B(2)s^2 + \dots$

set $\mu = \mathbb{E}[B] = G_B'(1)$

- We are interested in the behavior of Z_n as $n \to \infty$
- The extinction event E happens if $Z_n = 0$ for some n.

$$\Longrightarrow E = \bigcup_{n \geq 0} \{Z_n = 0\}$$

- Note that if $Z_n = 0$ then $0 = Z_{n+1} = Z_{n+2} = \dots$

so that $\{Z_n = 0\} \subset \{Z_{n+1} = 0\} \subset \dots$

$$\Longrightarrow \mathbb{P}(E) = \lim_{n \to \infty} \mathbb{P}(Z_n = 0)$$

Note that:

$$Z_{n+1} = B_{n,1} + B_{n,2} + \dots + B_{n,Z_n}$$

Walds formula $\Rightarrow$ $G_{Z_{n+1}}(s) = G_{Z_n}[G_B(s)]$

$$G_{n+1}(s) = G_{Z_{n+1}}(s) = G_n[G_B(s)]$$

and

$\mathbb{E}[Z_{n+1}] = \mu \mathbb{E}[Z_n]$ Since $Z_0 = 1$

$\mathbb{E}[Z_n] = \mu^n$ because $\mathbb{E}[Z_1] = \mu \cdot \overbrace{\mathbb{E}[Z_0]}^{1} = \mu$

$\mathbb{E}[Z_2] = \mu \cdot \mathbb{E}[Z_1] = \mu^2$

$\vdots$

$\Rightarrow$ if $\mu < 1$ then

$$\lim_{n\to\infty} \mathbb{E}[Z_n] = 0$$

- Now using Markov's inequality we deduce $P(Z_n > 0) = P(Z_n \geq 1) \leq \frac{1}{1}\mathbb{E}[Z_n] = \mu^n \xrightarrow[n\to\infty]{} 0$

$$P(Z_n = 0) = 1 - P(Z_n > 0) \xrightarrow[n\to\infty]{} 1$$

$\mathbb{P}(\mathbb{E}) = 1$ via implication at end of page 2

- If $\mu > 1$ then $\lim_{n\to\infty} \mathbb{E}[Z_n] = \lim_{n\to\infty} \mu^n = \infty$

⚠ However the probability that $Z_n = 0$ could be nonnegligible.

- Set $\rho_n = \mathbb{P}(Z_n = 0) = G_n(0)$. Note that $\rho_n = \mathbb{P}(Z_n=0) \le \mathbb{P}(Z_{n+1}=0) = \rho_{n+1}$

So ρ_n and $\underset{\forall n}{\rho_n \le 1} \Longrightarrow \exists \lim_{n\to\infty} \rho_n$

- Claim: If $p_0 = \mathbb{P}(B=0) > 0$ and $\mu = \mathbb{E}[B] > 1$ then $\rho_i = \lim_{n\to\infty} \rho_n \in (0,1)$

- Proof: $G_B(s) = p_0 + p_1(s) + p_2(s^2) + \dots$

$$\begin{cases} p_0 > 0 \\ G_B'(1) > 1 \end{cases}$$

- Note that $G_B(s)$ is convex $\Longrightarrow$ since $G_B'(1) > 1$ the graph of G_B near 1 is below the diagonal.

- Lemma: If $p_0 > 0$, and $\mu > 1$ then $s = G_B(s)$ has a unique solution $\rho \in (0,1)$

- Now $\rho_n = \mathbb{P}(Z_n=0) = G_n(0)$ and $G_{n+1}(s) = G_B\big(G_n(s)\big)$

$\Rightarrow \rho_1 = G_1(0) = G_B[G_0(0)] = G_B(0) < G_B(\rho) = \rho$

$\rho_2 = G_B(G_1(0)) = G_B(\rho_1) < G_B(\rho) = \rho$

$\vdots$

$\rho_n < \rho < 1 \quad \forall n \rightarrow \lim_{n\to\infty} \rho_n = \rho_\infty \le \rho < 1$

• Letting $n \to \infty$ in $\rho_{n+1} = G_B(\rho_n)$ we get $\rho_\infty = G_B(\rho_\infty)$, $\rho_\infty \in [0,1)$

$\Rightarrow \rho_\infty = \rho$ ∎

Ex $P(B=0) = \frac{1}{4}$, $P(B=1) = \frac{3}{8}$, $P(B=2) = \frac{3}{8}$, $P(B>2) = 0$

Then:

$$G_B(s) = \frac{2}{8} + \frac{3}{8}s + \frac{3}{8}s^2 = \frac{2+3s+3s^2}{8}$$

$p_0 = \frac{1}{4} > 0$, $\mu = \mathbb{E}[B] = 0 \cdot \frac{1}{4} + \left(1 \cdot \frac{3}{8}\right) + \left(2 \cdot \frac{3}{8}\right) = \frac{9}{8}$

$\mu > 1$

$G_B(s) = s \qquad \frac{2+3(s)+3(s^2)}{8} = s \quad \to \quad \boxed{s = \frac{2}{3}}$