Lecture 24

Generating Functions

- Recall, if X is a discrete r.v. with range $\mathcal{X} \subset \mathbb{N}_0 = \{0, 1, 2, \dots\}$ then $G_X(s) = \mathbb{P}(X=0) + \mathbb{P}(X=1)s + \mathbb{P}(X=2)s^2 + \dots \quad s \in [0,1]$

Rmk: $G_X(s) = \mathbb{E}[s^X]$

- Note that if $X_1, X_2, \dots, X_n$ are discrete r.v. with ranges $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ then $[\mathcal{X}_k \subset \mathbb{N}_0]$

$$s^{X_1 + X_2 + \dots + X_n} = s^{X_1} \cdot s^{X_2} \cdot \dots \cdot s^{X_n}$$

- If $X_1, X_2, \dots, X_n$ are <u>independent</u>, then $\mathbb{E}[s^{X_1 + X_2 + \dots + X_n}] = \mathbb{E}[s^{X_1} \cdot \dots \cdot s^{X_n}]$
$$\overset{\text{via independence}}{=} \mathbb{E}[s^{X_1}] \cdot \dots \cdot \mathbb{E}[s^{X_n}]$$

- We just proved:

<u>Prop</u>: If the discrete rand. var. $X_1, X_2, \dots, X_n$ are independent and their ranges contain in $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, then $G_{X_1 + X_2 + \dots + X_n}(s) = \prod_{i=1}^{n} G_{X_i}(s)$

$\boxed{\text{Ex}}$

a) Let $X_1, X_2, \ldots, X_n$ be <u>indep</u> and $X_K \sim Ber(p)$. $\forall K \in \{1, \ldots, n\}$

Set: $X = X_1 + X_2 + \ldots X_n$

then $X \sim Bin(n, p)$

Recall, that $G_{X_K}(s) = (1-p) + (p \cdot s)$

$$\Longrightarrow G_X(s) = [(1-p) + ps]^n$$

b) Let $X_1, \ldots, X_n$ be indep, and $X \sim Geom(p)$

Define $T_n = X_1 + \ldots + X_n$ then $T_n \sim NegBin(n, p)$

$$G_{X_i}(s) = \frac{(1-p)s}{1-ps} \quad \forall i \in \{1, \ldots, n\}$$

$$\Longrightarrow G_T(s) = \left[\frac{(1-p)s}{1-ps}\right]^n$$

c) Let $X_1, \ldots, X_n$ be indep, $X_K \sim Poi(\lambda_K)$

Note that $G_{X_K}(s) = e^{\lambda_K(s-1)}$

Set $X = X_1 + \ldots + X_n$. Then:

$$G_X(s) = \prod_{K=1}^{n} G_{X_K}(s) = \prod_{K=1}^{n} e^{\lambda_K(s-1)}$$

$$= e^{\lambda_1(s-1)} \cdot e^{\lambda_2(s-1)} \cdot \ldots \cdot e^{\lambda_n(s-1)} = e^{(\lambda_1 + \ldots + \lambda_n)(s-1)}$$

Thm: Wald's Formula

- Suppose that $X_1, \dots, X_n$ are i.i.d discr. r.v. with ranges contained in $\mathbb{N}_0$.
- Define by $G_X(s)$ their common pgf.

Let N be another discr. r.v. $N \perp X_i \; \forall i \in \{1, \dots, n\}$

with the range contained in $\mathbb{N}_0$, and pgf $G_N(s)$. Then the pgf of $S_N = X_1 + \dots + X_n$ is $G_{S_N}(s) = G_N(G_X(s))$. Moreover if $\mu = \mathbb{E}[X_i]$, $\sigma^2 = \text{var}[X_i]$, the

$$\mathbb{E}[S_N] = \mu \mathbb{E}[N] \text{ and } \text{var}[S_N] = \sigma^2 \mathbb{E}[N] + \mu^2 \text{var}[N]$$

Proof: Let $p_n = \mathbb{P}[N=n] \; \forall n \in \mathbb{N}_0$. Then $G_N(s) = \sum_{n=0}^{\infty} p_n s^n$

and $\forall n \in \mathbb{N}_0 \quad G_{S_n}(s) = [G_X(s)]^n$

Note that:

$$G_{S_N}(s) = \mathbb{E}[s^{S_N}] = \sum_{n=0}^{\infty} \mathbb{E}[s^{S_N} \mid N=n] \, \mathbb{P}(N=n) = \sum_{n=0}^{\infty} p_n \mathbb{E}[s^{S_n} \mid N=n]$$

- Since $N \perp\!\!\!\perp S_n$ we deduce

$$\mathbb{E}[s^{S_n} \mid N=n] = \mathbb{E}[s^{S_n}] = G_{S_n}(s)$$

- Hence

$$G_{S_N}(s) = \sum_{n=0}^{\infty} p_n G_{S_n}(s) = \sum_{n=0}^{\infty} p_n [G_X(s)]^n = G_N[G_X(s)]$$

$$\underbrace{\mathbb{E}[Y] = \sum_{x \in \mathcal{X}} \mathbb{E}[Y \mid X=x] \, p_X(x)}$$

↳ Generalization of the Law of Total Probability

$$\mathbb{E}[S_N] = G_{S_N}'(1)$$

$$= G_N'[G_X(1)] \cdot G_X'(1) = \underbrace{G_N'(1)}_{\mathbb{E}[N]} \cdot \underbrace{G_X'(1)}_{\mathbb{E}[X]}$$

$$= \mu\,\mathbb{E}[N]$$

- Next:

$$G_{S_N}''(1) = G_N''(1)\left(G_X'(1)\right)^2 + G_N'(1)\,G_X''(1)$$

$$= \mu^2 G_N''(1) + \mathbb{E}[N]\,G_X''(1)$$

use this:

$$G_X''(1) = \mathrm{var}[X] + \mu^2 - \mu$$

to conclude $\square$