# Lecture 23

## Warm up Questions :

1. Let $X$ be a d.r.v. with $\mathcal{X} = \{-1, 0, 2\}$ and pmf $p$ given by $P(-1) = \frac{3}{11}$, $P(0) = \frac{7}{11}$, and $P(2) = \frac{1}{11}$

   Find $\mathbb{E}[X]$.

$$\mathbb{E}[X] = (-1)P(X=-1) + 0\,P(X=0) + 2P(X=2)$$

2. Let $\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$. Define $\text{cov}[X,Y]$

$$\text{cov}[X,Y] \stackrel{def}{=} \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \stackrel{cor}{=} \mathbb{E}[XY] - \mu_X \mu_Y$$

3) True or False

a) $\text{var}[X] = \mathbb{E}[X - \mu_X]$ False $\text{var}[X] = \mathbb{E}[(X-\mu_X)^2]$

b) $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ True (linearity of expectation)

c) $\mathbb{E}[f(X,Y)] = \sum_{(x,y)\in \mathcal{X}\times\mathcal{Y}} f(x,y)\,p(x,y)$ True (law of subconscious statistician)

d) $\mathbb{E}[f(X,Y)] = \sum_{(x,y)\in \mathcal{X}\times\mathcal{Y}} f(x,y)\,p(x)p(y)$ False, if $X \perp\!\!\!\perp Y$ then true

e) $\mathbb{E}[f(X)\,g(Y)] = \mathbb{E}[f(X)]\,\mathbb{E}[g(Y)]$ False, if $X \perp\!\!\!\perp Y$ then true

$\hookrightarrow \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ is true if $X$ and $Y$ are uncorrelated, remember independence implies uncorrelation

f) $\text{var}[X+Y] = \text{var}[X] + \text{var}[Y]$ False, $\text{var}[X+Y] = \text{var}[X] + \text{var}[Y] + 2\,\text{cov}[X,Y]$

also $\text{var}[aX+bY] = a^2\,\text{var}[X] + b^2\,\text{var}[Y] + 2ab\,\text{cov}[X,Y]$

$a, b \in \mathbb{R}$

**Thm**: Law of Subconscious Statistician. If $(X,Y)$ is a continuous random vector with joint pdf $p(x,y)$ and $f(x,y)$ is a function of two variables, then

$$\mathbb{E}[f(X,Y)] = \iint_{\mathbb{R}^2} f(x,y)\, p(x,y)\, dx\, dy$$

**Proof**:

$$\mathbb{E}[(X+Y)] = \iint_{\mathbb{R}^2} (x+y)\, p(x,y)\, dx\, dy$$

similarly

$$= \underbrace{\iint_{\mathbb{R}^2} x\, p(x,y)\, dx\, dy}_{} + \iint_{\mathbb{R}^2} y\, p(x,y)\, dx\, dy = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$= \int_{\mathbb{R}} \Big( \underbrace{\int_{\mathbb{R}} p(x,y)\, dy}_{= p(x)} \Big)\, x\, dx = \int_{\mathbb{R}} x\, p(x)\, dx = \mathbb{E}[X]$$

**Cor**: Linearity of Expectation. If $(X,Y)$ is a continuous random vector then $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

**Proof**:

$$X \perp\!\!\!\perp Y \rightarrow p(x,y) = p_X(x)\, p_Y(y)$$

$$\mathbb{E}[f(X)g(Y)] = \iint_{\mathbb{R}^2} f(x)\, g(y)\, p(x,y)\, dx\, dy$$

$$= \iint_{\mathbb{R}^2} f(x)\, p_X(x)\, g(y)\, p_Y(y)\, dx\, dy \quad \text{(use Fubini thm)}$$

$$= \Big( \int_{\mathbb{R}} f(x)\, p_X(x)\, dx \Big) \Big( \int_{\mathbb{R}} g(y)\, p_Y(y)\, dy \Big) = \mathbb{E}[f(X)]\, \mathbb{E}[g(Y)]$$

$\boxed{\text{Ex}}$ Let $T_0 \sim \mathrm{Exp}(\lambda_0)$, $T_1 \sim \mathrm{Exp}(\lambda_1)$

$\left[\, \mathbb{P}(T_{0,1} \geq t) = e^{-\lambda_{0,1} t} \,\right]$ Assume $T_0 \perp\!\!\!\perp T_1$

Denote:

$$T = \min(T_0, T_1)$$

set: $N = \begin{cases} 1, & T = T_1 \\ 0, & T = T_0 \end{cases}$

<u>claim</u>: $T \sim \mathrm{Exp}(\lambda_0 + \lambda_1)$

Indeed $\mathbb{P}(T \geq t) = \mathbb{P}(T_0 \geq t, T_1 \geq t)$

$$= \mathbb{P}(T_0 \geq t) \cdot \mathbb{P}(T_1 \geq t)$$

$$= e^{-\lambda_0 t} \cdot e^{-\lambda_1 t} = e^{-(\lambda_0 + \lambda_1)t}$$

$\blacksquare$ claim

• Let us compute $\mathbb{P}(N=0)$

$$\mathbb{P}(N=0, T \geq t) = \mathbb{P}(T_1 \geq T_0 \geq t)$$

$$= \iint_{t \leq x_0 \leq x_1} \lambda_0 e^{-\lambda_0 x_0} \lambda_1 e^{-\lambda x_1} \, dx_0 \, dx_1$$

$$= \lambda_0 \lambda_1 \int_t^\infty \left( \int_{x_0}^\infty e^{-\lambda_1 x_1} dx_1 \right) e^{-\lambda_0 x_0} dx_0$$

$$= \lambda_0 \int_t^\infty e^{-(\lambda_0 + \lambda_1) x_0} dx_0 = \frac{\lambda_0}{\lambda_0 + \lambda_1} e^{-(\lambda_0 + \lambda_1)t} = \frac{\lambda_0}{\lambda_0 + \lambda_1} \mathbb{P}(T \geq t)$$

$$\mathbb{P}(N=1, T \geq t) = \frac{\lambda_1}{\lambda_0 + \lambda_1} \mathbb{P}(T \geq t) = \mathbb{P}(N=1)\,\mathbb{P}(T \geq t)$$

$\mathbb{P}(N=0, T=0) = \mathbb{P}(N=0) = \frac{\lambda_0}{\lambda_1 + \lambda_0}$ $\Longrightarrow$ $T \perp\!\!\!\perp N$

$\mathbb{P}(N=1) = \frac{\lambda_1}{\lambda_1 + \lambda_0}$

<u>Def</u>: Suppose that $(X, Y)$ is a continuous rand. vector. Let $\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$. Then the covariance of $(X, Y)$ is the number

$$\mathrm{cov}[X, Y] = \mathbb{E}\left[(X - \mu_X)(Y - \mu_Y)\right]$$

- And the correlation coefficient of $(X, Y)$ is

$$\rho[X, Y] = \frac{\mathrm{cov}[X, Y]}{\sqrt{\mathrm{var}[X] \cdot \mathrm{var}[Y]}}$$

<u>Rmk</u>: If $p(x, y)$ is the joint pdf, then

$$\mathrm{cov}[X, Y] = \mathbb{E}[XY] - \mu_X \mu_Y$$

<u>Cor</u>: $X \perp\!\!\!\perp Y \Longrightarrow \mathrm{cov}[X, Y] = 0$ and $\mathrm{var}[X + Y] = \mathrm{var}[X] + \mathrm{var}[Y]$

## Multi-dimensional continuous random vectors

<u>Definition</u>: Suppose that $X_1, \ldots, X_n$ are $n$ random variables. We say that the random vector $(X_1, \ldots, X_n)$ is continuous if there exists a function of $n$ {fill in rest of slides}