Lecture 22

Multivariate Continuous Distributions

① Two-dimensional cont. rand. vectors X, Y : two rand. var. defined on (S, P). We say that X and Y are jointly continuous if $\exists\ p: \mathbb{R}^2 \to \mathbb{R}$ s.t. $P(X \le x, Y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{x} p(s,t)\,ds\,dt$

Rmk: $\Leftrightarrow\ \forall\ B \subset \mathbb{R}^2$

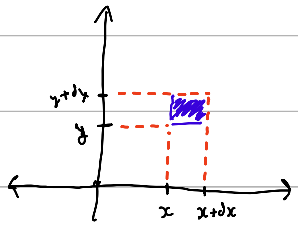
$$P[(X,Y) \in B] = \int_B p(x,y)\,dx\,dy$$

Def: p is called joint pdf of X and Y, (X,Y) - continuous random vector

joint cdf: $F_{X,Y}: \mathbb{R}^2 \to [0,1]$

$$F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{x} p(s,t)\,ds\,dt$$

Rmk: $p(x,y)\,dx\,dy = P(X \in [x, x+dx], Y \in [y, y+dy])$ or "$p(x)dx = P(X \in [x+dz])$



Prop: Suppose that X and Y are jointly continuous rand. var. with joint pdf p and joint cdf $F(x,y)$. Then X and Y are continuous rand. var. with pdf p_X and p_Y (marginals) given by

$$p_X(x) = \int_{\mathbb{R}} p(x,y)\,dy \qquad p_Y(y) = \int_{\mathbb{R}} p(x,y)\,dx$$

• **Moreover:**

$$p(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

• **Rmk:** $f: \mathbb{R}^2 \to \mathbb{R}$ is the joint pdf of a continuous random vector (X,Y) i.f.f. :

$$\begin{cases} f(x,y) \geq 0 \quad \forall x,y \\ \int_{\mathbb{R}^2} p(x,y)\,dx\,dy = 1 \end{cases}$$

• **Ex:** (Independent rand var)

Recall: $X \perp\!\!\!\perp Y$ i.f.f. $\forall A, B \subset \mathbb{R}$

$$P(X \in A, Y \in B) = p(X \in A)\, p(Y \in B)$$

• If X and Y are continuous rand. var. with pdf's p_x and p_y then

$$X \perp\!\!\!\perp Y \iff p(x,y) = p_X(x)\, p_Y(y) \quad \forall x, y \in \mathbb{R}$$

• $X \perp\!\!\!\perp Y$, $S := X + Y$

$$P(S \leq s) = P(X + Y \leq s)$$

$$= \int_{x+y \leq s} p_X(x)\, p_Y(y)\, dx\, dy$$

$$= \int_{\mathbb{R}} \underbrace{\left(\int_{-\infty}^{s-x} p_Y(y)\, dy \right)}_{P(Y \leq s-x)} p_x(x)\, dx$$

$$= \int_{\mathbb{R}} P(Y \leq s-x)\, p_X(x)\, dx$$

Derivating w.r.t. s we get :

$$p_{X+Y}(s) = \int_{\mathbb{R}} p_Y(s-x)\, p_X(x)\, dx$$

$\boxed{\text{Ex}}$: Uniform Distribution on a planar region

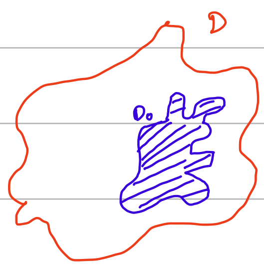
- Let $D \subset \mathbb{R}^2$, Area(D) is finite .
- Then the uniform distribution on D is described by :

$$p(x,y) = \frac{1}{\text{Area}(D)} I_D(x,y)$$

$$I_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \notin D \\ 1 & \text{if } (x,y) \in D \end{cases}$$

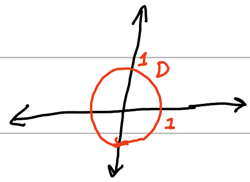
- Thus , $\forall\, D_0 \subset D$

$$P((x,y) \in D_0) = \frac{\text{Area}(D_0)}{\text{Area}(D)}$$



- Rmk : The specific location of D_0 in D, or the slope of D_0 play NO role in this case.

• Let $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$



Area$(D) = \pi$, so

$$p(x,y) = \frac{1}{\pi} I_D(x,y)$$

• Let us compute the marginals $p_X(x)$, $p_Y(y)$:

- Note that if $|x| > 1$ we have $p_X(x) = 0$

• for $|x| \leq 1$:

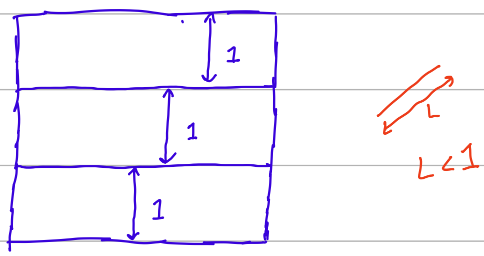
$$p_X(x) = \frac{1}{\pi} \int_{\mathbb{R}} I_D(x,y)\,dy = \frac{1}{\pi} \int_{|y| \leq \sqrt{1-x^2}} 1\,dy$$

$$= \frac{2}{\pi} \sqrt{1-x^2}$$

$p_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}$ if $|y| \leq 1$, and $p_Y(y) = 0$ if $|y| > 1$.

• Ex : Buffon needle problem

• Assume we have a plane ruled by parallel lines at unit distance. We are placing a needle of length L.
What is the probability $p = p(L)$ that the needle intersects on of these lines ?



• The distance between the center of the needle and the closest line is y.

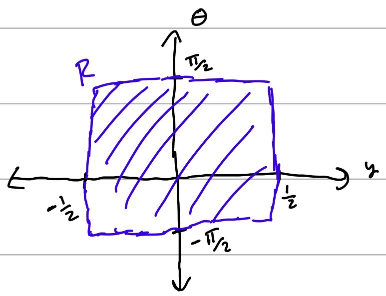
↳ If the closest line is above the center point then $y \geq 0$. If other case then $y > 0$.

↳ So $y \in [-\frac{1}{2}, \frac{1}{2}]$

- θ is the angle between the needle and the vertical line drawn through the center of the needle.

  ↳ so $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

- The point (y, θ) is uniformly distributed in $R = \{(y,\theta) : -\frac{1}{2} \le y \le \frac{1}{2}, \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$



- Note that the joint pdf of (y, θ) is :

$$f(y,\theta) = \frac{1}{\text{Area}(R)} I_R(y,\theta) = \frac{1}{\pi} I_R(y,\theta)$$

- $$P(L) = \mathbb{P}(|y| \le \tfrac{1}{2} L \cos\theta) = \frac{1}{\pi} \iint_{|y| \le \frac{L\cos\theta}{2}} I_R(y,\theta)\, dy\, d\theta = \frac{2L}{\pi}$$