Lecture 21

Conditioning

- Let (P, S) be a probability space and $E \subseteq S$ be an event. Then E is a probability space with probability function $P(- | E)$.
  - ↳ "conditioning on E"

- Given a r.v. X, let $X|E$ denote the restriction of X to E.

- If X is discrete, the pmf of $X|E$ (called the conditional pmf of X given E) is:

$$P_{X|E}(x) = P(X = x | E) = \frac{P(X = x \cap E)}{P(E)}$$

- If X and Y are discrete rv's, the conditional pmf of X given $\overbrace{Y = y}^{\text{Event } E}$ is given by

$$P_{X|Y=y}(x) = \frac{P(X = x \mid Y = y)}{P(Y = y)} = \frac{P_{X,Y}(x, y)}{P_Y(y)}$$

Ex Let X be a uniformly random element $\{1, 2, 3, 4\}$

Let Y be a uniformly random element $\{1, 2, \ldots, x\}$

(i) What is the joint pmf $P_{X,Y}$?

marginals

$y \backslash x$	1	2	3	4	P_Y
4	0	0	0	1/16	1/16
3	0	0	1/12	1/16	7/48
2	0	1/8	1/12	1/16	13/48
1	1/4	1/8	1/12	1/16	25/48
P_X	1/4	1/4	1/4	1/4	

ii) Whats the pmf $x|y=2$ so range of $X = \{2, 3, 4\}$

$$P_{X|y=2}(2) = \frac{P_{X,Y}(2,2)}{P_Y(2)} = \frac{1/8}{13/48} = \frac{6}{13}$$

$$P_{X|y=3}(3) = \frac{P_{X,Y}(3,2)}{P_Y(2)} = \frac{1/12}{13/48} = 4/13$$

$$P_{X|y=4}(4) = \frac{P_{X,Y}(4,2)}{P_Y(2)} = \frac{1/16}{13/48} = 3/13$$

Corolary:

Let X and Y be discrete rv's. Then X and Y are independent i.f.f.

$$X \mid Y = y \sim X \quad \text{for all } y \text{ in range}(Y)$$

i.e. $P_{X|Y=y} = P_X$ for all y in range(Y)

Note: In our example $P_{X|y=2} \neq P_X$ $\therefore$ X and Y are dependant

- We also have a law of total probability for expectation:

Prop 3.31 (law of total expectation)

↳ Let X be a r.v. and let $B_1, B_2, \ldots$ be a partition of the sample space S.

- $B_1 \cup B_2 \cup \ldots = S$
- $B_i \cap B_j = \emptyset \;\; \forall\, i \neq j$

↳ Then:

$$\mathbb{E}[X] = \sum_{k \geq 1} \mathbb{E}[X \mid B_k]\, \mathbb{P}(B_k)$$

• $\boxed{\text{Ex}}$ We roll a fair 4-sided die until we get a 4.

Let R be the number of rolls. We then flip a fair coin R times and let N be the number of heads. What is $\mathbb{E}[N]$?

Use Law of total expectation with B_k = (event $R=k$) for all $k \geq 1$

$$\mathbb{E}[N] = \sum_{k\geq 1} \underbrace{\mathbb{E}[\overbrace{N \mid R=k}^{\text{Bin}(k,\,1/2)}]}_{\frac{k}{2}} \; \underbrace{\mathbb{P}(\overbrace{R}^{\text{Geom}(\frac{1}{4})}=k)}_{(\frac{1}{4})(\frac{3}{4})^{k-1}}$$

$$= \sum_{k=1}^{\infty} \frac{k}{2}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{k-1}$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} k \cdot \underbrace{\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{k-1}}_{\mathbb{P}(R=k)}$$

$$= \frac{1}{2}\,\mathbb{E}[R] = \frac{1}{2} \cdot \frac{1}{1/4} = 2$$

• Given r.v.'s X and Y, we can define the conditional random variable

$\mathbb{E}[Y \mid X]$ on the sample space range(X)

function whose value at x is $\mathbb{E}[y \mid X=x]$

• The law of total expectation states that:

$\mathbb{E}\big[\mathbb{E}[Y \mid X]\big] = \mathbb{E}[Y]$ "law of subconscious statistician"

$\boxed{\text{Ex}}$ Coin Patterns

- Repeatedly flip a fair coin. Given a pattern p (e.g. $p=H$ or $p=HT$), let W_p be the number of flips until we obtain the pattern p on cosecutive flips. We want to find $\mathbb{E}[W_p]$

e.g. $\underset{\substack{\| \\ \text{Geom}\left(\frac{1}{2}\right)}}{\mathbb{E}[W_H]} = \frac{1}{1/2} = 2.$ $\quad \mathbb{E}[W_T] = \frac{1}{2}$

$\boxed{\text{e.g.}}$ $$\overbrace{\mathbb{E}[W_{HH}]}^{t} = \underbrace{\mathbb{E}[W_{HH} \mid 1^{st} \text{ flip } H]}\, \underbrace{P(1^{st} \text{ flip } H)}_{\frac{1}{2}} + \underbrace{\mathbb{E}[W_{HH} \mid 1^{st} \text{ flip } T]}_{1+t}\, \underbrace{P(1^{st} \text{ flip } T)}_{\frac{1}{2}}$$

Law of total expectation

$$= \underbrace{\mathbb{E}[W_{HH} \mid \text{flips } HH]}_{2} \cdot \frac{1}{2} + \underbrace{\mathbb{E}[W_{HH} \mid \text{flips } HT]}_{2+t} \cdot \frac{1}{2}$$

So $\quad t = \left[2\left(\frac{1}{2}\right) + (2+t)\frac{1}{2}\right] \cdot \left(\frac{1}{2}\right) + (1+t) \cdot \frac{1}{2} \implies t = 6$

$\boxed{\text{e.g.}}$ $\quad \mathbb{E}[W_{HT}] = 4$

$\boxed{\text{e.g}}$ $\quad \mathbb{E}[W_{HTHT}] = 20 \quad = \quad 2^4 + 2^2$

$\boxed{\text{e.g}}$ $\quad \mathbb{E}[W_{HHHH}] = 30 \quad = \quad 2^{4} + 2^{3} + 2^{2} + 2^{1}$

H H H H