Lecture 2

Inclusion - Exclusion Formula

Prop: Assume we have probability space (S, P), then the following holds:

(i) $P(A^c) = 1 - P(A), \quad \forall A \subset S$

this is $A - B$ (set subtraction)

(ii) $P(A \setminus B) = P(A) - P(A \cap B) \quad \forall A, B \subset S$

(iii) Inclusion Exclusion Principle:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \forall A, B \subset S$$

(iv) De Morgan

$$P(A^c \cap B^c) = 1 - P(A \cup B)$$

(v) $\forall A, B \subset S, \quad A \subset B \rightarrow P(A) \leq P(B)$

Proof:

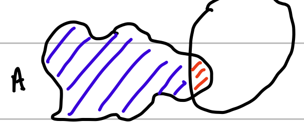
(i) $A \cup A^c = S, \quad A \cap A^c = \emptyset$

$P(A \cup A^c) = P(A) + P(A^c) = P(S) = 1$

$\rightarrow P(A^c) = 1 - P(A)$

(ii) $A = (A \setminus B) \cup (A \cap B)$

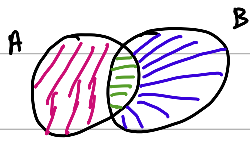
Disjoint Union



$(A \setminus B) \cap (A \cap B) = \emptyset$

$\rightarrow P(A \setminus B) \cup P(A \cap B) = P(A \setminus B) + P(A \cap B)$

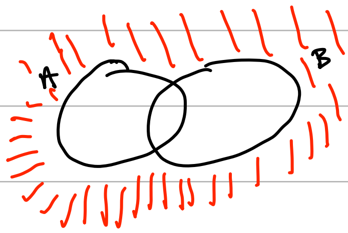
(iii) $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$



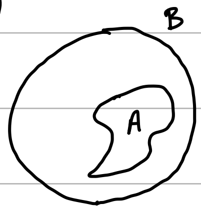
$\rightarrow P(A \cup B) = \underbrace{P(A \setminus B)}_{P(A) - P(A \cap B)} + P(A \cap B) + \underbrace{P(B \setminus A)}_{P(B) - P(B \cap A)}$

$= P(A) + P(B) - P(A \cap B)$

(iv) $A^c \cap B^c = \underline{(A \cup B)^c} \rightarrow P((A \cup B)^c) = 1 - P(A \cup B)$



(v)



$B = A \cup (B \setminus A)$

$A \cap (B \setminus A) = \varnothing$

$P(B) = P(A) + \underbrace{P(B \setminus A)}_{\geq 0} \rightarrow P(B) \geq P(A)$

Problem set (Lecture 2) Question 1

A = rain on Saturday | $P(A) = \frac{1}{2}$

B = rain on Sunday | $P(B) = \frac{1}{2}$

Can one conclude that the probability of rain (at some point) during the weekend is 100%?

$P(A \cup B) = 1$?

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = \frac{1}{2} + \frac{1}{2} - (\frac{1}{2})(\frac{1}{2}) = 0.75$

$\boxed{\text{Ans: Not ALWAYS.}}$

Inclusion - Exclusion Formula for 3 events : $A_1, A_2, A_3 \subset S$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3)$$
$$- P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

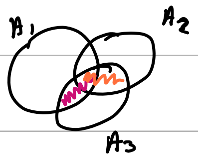
Proof:

$$P(\underbrace{(A_1 \cup A_2)}_{A} \cup \underbrace{A_3}_{B}) = P(\underbrace{A_1 \cup A_2}_{A}) + P(\underbrace{A_3}_{B}) - P(\underbrace{(A_1 \cup A_2) \cap A_3}_{A \cap B})$$

$$= P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P((A_1 \cup A_2) \cap A_3)$$

$$(A_1 \cup A_2) \cap A_3 = (A_1 \cap A_3) \cup (A_2 \cap A_3)$$



$$P((A_1 \cup A_2) \cap A_3) = P(A_1 \cap A_3) + P(A_2 \cap A_3) - P(\underbrace{(A_1 \cap A_3) \cap (A_2 \cap A_3)}_{A_1 \cap A_2 \cap A_3})$$

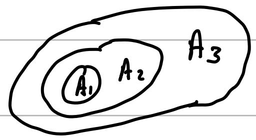
∎

· General Inclusion - Exclusion Principle

Given: (S, P) $A_1, A_2, \ldots, A_n \subset S$

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{k=1}^{n} P(A_k) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots$$

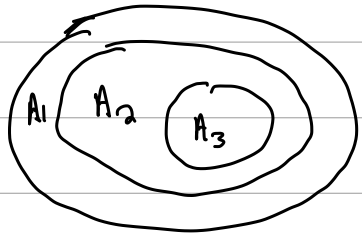
Def: A sequence of events is called <u>increasing</u> if $A_1 \subset A_2 \subset A_3 \subset \dots$



Define:

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

Def: A sequence of events is called <u>decreasing</u> if $A_1 \supset A_2 \supset A_3 \supset \dots$



Define:

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Prop: If $A_1, A_2, \dots, A_n$ is a sequence of incr/decr events then:

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

Proof: follows from countable additivity of P

Finite Sample Spaces and Counting

$S = \{s_1, s_2, \dots, s_n\}$ $\qquad$ P_{unif}: uniform distribution on S

$$P_{unif}(A) = \frac{|A|}{n} = \frac{\text{the number of favorable outcomes}}{\text{the number of all outcomes}}$$

Problem Set (Lecture 2) Problem 2

- Q: Arbitrary family with 3 children, what is prob that exactly 2 of the children are right handed. (Assume right handedness and left handedness are equally likely)

Outcomes: RRR, $\underbrace{\text{LRR, RLR, RRL}}_{\text{favorable outcomes}}$, RLL, LRL, LLR, LLL

So prob is $3/8$

Thm: Multiplication Principle

- Perform in order r experiments such that the number of possible outcomes at k^{th} experiment is n_k. Then the number of possible outcomes of this ordered sequence of experiments is $n_1 \cdot n_2 \cdot \ldots \cdot n_k$

$\boxed{\text{Ex}}$ Toss a coin twice, how many outcomes?

H H
T T

$2 \times 2 = 4$

$\boxed{\text{Ex}}$ Roll a die 3 times, how many outcomes

6^3

$\boxed{\text{Ex}}$ Assume choosing outfit. Outfit has a shirt, short, and shoes. You have 9 shirts, 7 shorts, and 5 shoes. How many possible outfits?

$9 \times 7 \times 5$

Problem Set (Lecture 2) - Question 3.

$P(H) = 2P(T)$, $P(T) = P(T)$, $P(Edge) = 0.1$

$P(H \cup T) = 1 - P(Edge) = 0.9$

$P(H \cup T) = P(H) + P(T) - \cancel{P(H \cap T)}^{\,0}$

$P(H \cup T) = 2P(T) + P(T)$

$P(H \cup T) = 3P(T)$

$0.9 = 3P(T)$

$P(T) = 0.3$

so $1 = P(H) + P(T) + P(Edge)$

$1 = P(H) + 0.3 + 0.1$

$\boxed{P(H) = 0.6}$