Lecture 19

Def: The range of a discrete random vector (X, Y) consists of the pairs (x, y) s.t. $\mathbb{P}(X = x, Y = y) \neq 0$.

A discrete random vector (X, Y) with range R is said to be uniformly distributed if the restriction to R of the joint pmf is a constant function.

Thm: The Law of Subconscious Statitician:

Suppose that X, Y are two discrete random variables with ranges $\mathcal{X}$ and $\mathcal{Y}$. If $p(x, y)$ is the joint pmf of the random vector (X, Y) and $g(x, y)$ is a function of two variables defined on $\mathcal{X} \times \mathcal{Y}$, then

$$\mathbb{E}[g(X, Y)] = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} g(x, y)\, p(x, y).$$

Cor: Linearity of Expectation

a) Suppose X, Y are 2 discrete r.v then $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

b) Let $X_1, \dots, X_n$ be discrete r.v. and $c_1, \dots, c_n \in \mathbb{R}$. Then

$$\mathbb{E}[c_1 X_1 + \dots + c_n X_n] = c_1 \mathbb{E}[X_1] + \dots + c_n \mathbb{E}[X_n]$$

Rmk: Random Variables $X_1, X_2, \dots, X_n$ need NOT to be independent.

Def: Covariance and Correlation:

Let X, Y be discrete r.v. with ranges $\mathcal{X}$ and $\mathcal{Y}$.

Let $p(x,y)$ be the joint pmf of (X,Y). Assume that X, Y are 2 integrable, $\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$

i) The covariance of X, Y:

$$\begin{aligned} \operatorname{cov}[X,Y] &= \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \mathbb{E}[XY] - \mu_Y \mathbb{E}[X] - \mu_X \mathbb{E}[Y] + \mu_X \mu_Y \\ &= \mathbb{E}[XY] - \mu_X \mu_Y \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \end{aligned}$$

ii) The correlation coefficient of X, Y:

$$\rho[X,Y] = \frac{\operatorname{cov}[X,Y]}{\sqrt{\operatorname{var}[X]\operatorname{var}[Y]}}$$

rmk: $\rho \in [-1, 1]$

iii) The discrete r.v. X, Y are called uncorrelated if:

$$\operatorname{cov}[X,Y] = \rho[X,Y] = 0$$

Prop: X, Y - discr r.v. , $a, b \in \mathbb{R}$ then

$$\mathrm{var}[aX+bY] = a^2 \mathrm{var}[X] + b^2 \mathrm{var}[Y] + 2ab\,\mathrm{cov}[X,Y]$$

Proof: Let $\mu_x = \mathbb{E}[X]$, $\mu_y = \mathbb{E}[Y]$

$$\mu \overset{\text{def}}{=} \mathbb{E}[aX+bY] = a\mu_x + b\mu_y$$

Then $(aX+bY) - \mu = aX + bY - a\mu_x - b\mu_y$

$$= a(X-\mu_x) + b(Y-\mu_y)$$

$$\mathrm{var}[aX+bY] = \mathbb{E}\left[(aX+bY-\mu)^2\right]$$

$$\mathbb{E}\left[(aX+bY-\mu)^2\right] = \mathbb{E}\left[\left(a(X-\mu_x)+b(Y-\mu_y)\right)^2\right]$$

$$= \mathbb{E}\left[a^2(X-\mu_x)^2 + b^2(Y-\mu_y)^2 + 2ab(X-\mu_x)(Y-\mu_y)\right]$$

$$= a^2 \underbrace{\mathbb{E}\left[(X-\mu_x)^2\right]}_{\mathrm{var}[X]} + b^2 \underbrace{\mathbb{E}\left[(Y-\mu_y)^2\right]}_{\mathrm{var}[Y]} + 2ab\,\mathrm{cov}[X,Y]$$

$\blacksquare$

Cor: If X and Y are <u>uncorrelated</u>, the $\mathrm{var}[X+Y] = \mathrm{var}[X] + \mathrm{var}[Y]$

$\boxed{\text{Ex}}$: (S,P) - prob space ; $A_1, A_2 \subset S$ with $P(A_i) = p_i$, $(i=1,2)$

The indicator functions:

$$I_{A_i}(s) = \begin{cases} 1, & s \in A_i \\ 0, & s \notin A_i \end{cases}$$

$I_{A_i} \sim Ber(p_i)$

Note that $I_{A_1} \cdot I_{A_2} = I_{A_1 \cap A_2}$

and $\mathbb{E}[I_{A_i}] = p_i$

$$cov[I_{A_1}, I_{A_2}] = \mathbb{E}[I_{A_1} I_{A_2}] - p_1 p_2 = P(A_1 \cap A_2) - p(A_1)p(A_2)$$

★ • I_{A_1} and I_{A_2} are uncorrelated i.f.f. the events A_1 and A_2 are independent.

Prop: Suppose that X, Y are independent discrete random variables. Then for any two functions f and g, the r.v. $f(X)$ and $g(Y)$ are independent and $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)] \cdot \mathbb{E}[g(Y)]$

• In particular $X \perp\!\!\!\perp Y \Rightarrow cov[X,Y] = 0$

joint pmf of (x,y)

Proof: $X \perp\!\!\!\perp Y \implies p(x,y) = p_X(x)\, p_Y(y)$

$$\mathbb{E}[f(X)g(Y)] = \sum_{\substack{x \in X \\ y \in Y}} f(x)\, g(y)\, p(x,y)$$

$$h(X,Y) = f(X)g(Y) = \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} f(x)\, g(y)\, p_X(x)\, p_Y(y)$$

$$= \left(\sum_{x \in \mathcal{X}} f(x)\, p_X(x)\right)\left(\sum_{y \in \mathcal{Y}} g(y)\, p_Y(y)\right)$$

$$= \mathbb{E}[f(X)] \cdot \mathbb{E}[g(Y)]$$

$\blacksquare$

Cor: If $X_1, X_2, \dots, X_n$ are INDEPENDENT random variables, then

$$\mathrm{var}[X_1 + \dots + X_n] = \mathrm{var}[X_1] + \dots + \mathrm{var}[X_n]$$

<u>Ex</u>: Consider discrete r.v. X and Y with joint pmf

$Y \downarrow$			
1	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{1}{20}$
0	$\frac{2}{20}$	$\frac{8}{20}$	$\frac{2}{20}$
-1	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{1}{20}$
$p(x,y)$	-1	0	1

$\leftarrow X$

$P(X=-1) = \frac{1}{20} + \frac{2}{20} + \frac{1}{20} = \frac{4}{20} = \frac{1}{5}$

$P(Y=-1) = \frac{1}{5}$

$P(X=1) = P(Y=1) = \frac{1}{5}$

$P(X=0) = \frac{2+8+2}{20} = \frac{3}{5}$

$P(Y=0) = \frac{2+8+2}{20} = \frac{3}{5}$

$\mathbb{E}[X] = (-1)p(X=-1) + (0)p(X=0) + 1p(X=1)$

$= (-1)(\frac{1}{5}) + 0 + 1(\frac{1}{5}) = 0$

$\mathbb{E}[Y] = (-1)P(Y=-1) + (0)P(Y=0) + (1)P(Y=1) = 0$

$\Longrightarrow \operatorname{cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[XY]$

$\mathbb{E}[XY] = 1 \cdot [P(X=1, Y=1) + P(X=-1, Y=-1)]$

$+ 0\,[P(X=0, Y=1) + P(X=0, Y=0) + \ldots + P(X=1, Y=0) + \ldots]$

$+ (-1)\,[P(X=1, Y=-1) + P(X=-1, Y=1)]$

$= 0$

but they are not independent.