Lecture 18

• Functions of Continuous Random Variables

• $X \sim$ cont r.v. with pdf p, $g: \mathbb{R} \to \mathbb{R}$ is a function then $g(X)$ is also a r.v.

Ex: $g(x) = x$ then $g(X) = -X$ and pdf of $g(X)$ is $p(-y)$

cdf: $F_{g(x)}(y) = P[g(X) \leq y] = P(-X \leq y) = P(X \geq -y) = \int_{-y}^{\infty} p(x)\,dx \Rightarrow F_{-X}'(y) = p(-y)$ ☒

Rmk: Let a function $g(x) = ax + b$ with $a > 0$, X be a continuous r.v. Then $Y = g(X)$ is also a continuous r.v.

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$$
$$= P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

Derviating with respect to y:

$$q_y = F'_x(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a} p_X\left(\frac{y-b}{a}\right)$$

• Gaussian random variables:

• Let $X \sim N(0,1)$ then $\forall\, \sigma > 0$, $\mu \in \mathbb{R}$

the r.v. $Y = \sigma x + \mu$ is Gaussian $N(\mu, \sigma^2)$.

• Indeed, the pdf of X is $p(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$\Rightarrow$ the pdf of Y $\quad p_Y(y) = \frac{1}{\sigma} p_X\left(\frac{y-\mu}{\sigma}\right) \cdot \frac{(y-\mu)^2}{2\sigma^2} = \frac{1}{\sigma\sqrt{2\pi}} e \Rightarrow Y \sim N(\mu, \sigma^2)$

Note that

$$\mathbb{E}[Y] = \sigma \mathbb{E}[X] + \mu = 6.0 + \mu = \mu$$

$$\text{var } \mathbb{E}[X] = \sigma^2 (\text{var}[X]) = \sigma^2$$

Conversely, $Y \sim N(\mu, \sigma^2)$ then $X = \frac{1}{\sigma}(Y - \mu) \sim N(0,1)$

Discrete random vectors:

- Let X, Y be two discrete r.v. defined on the same probability space (S, P).

  $X, Y : S \to \mathbb{R}$

  Random vector

  $$(X, Y) : S \to \mathbb{R}^2$$

The range of X is $\mathcal{X}$

The range of Y is $\mathcal{Y}$

then the range (X, Y) contained in $\mathcal{X} \times \mathcal{Y}$

(X, Y) is defined by the joint probability mass function (joint pmf)

$$p : \mathcal{X} \times \mathcal{Y} \to [0, 1]$$

$$p(x, y) = P(X = x, Y = y)$$

Note that $\sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p(x, y) = 1$

The pmf of X and pmf of Y are called marginal pmf's or the marginals of the discrete random vector (X, Y).

Ex Let X and Y be discrete random variables with ranges $\mathcal{X}$ and $\mathcal{Y}$.

Consider their pmf's:

$$p_X : \mathcal{X} \to [0,1] \qquad p_Y : \mathcal{Y} \to [0,1]$$

The r.v.'s X and Y are independent i.f.f. $P(X=x, Y=y) = P_X(X=x)\, P_Y(Y=y)$

Thus, $X \perp\!\!\!\perp Y$ i.f.f.

$$P_{(X,Y)}(x,y) = P_X(x)\, P_Y(y)$$

Prop: Marginals

- Suppose X and Y are two discrete r.v. with ranges $\mathcal{X}$ and $\mathcal{Y}$ and pmf's P_X and P_Y. If $p(x,y)$ is the joint pmf of (X,Y) then

$$P_X(x) = \sum_{y \in \mathcal{Y}} p(x,y) \quad \text{and} \quad P_Y(y) = \sum_{x \in \mathcal{X}} p(x,y)$$

Ex 2 cards out of deck of 52

X = # of hearts $\quad \mathcal{X} = \{0, 1, 2\}$

Y = # of Queens $\quad \mathcal{Y} = \{0, 1, 2\}$

Y \ X	0	1	2
0	$\binom{36}{2}/\binom{52}{2}$	$12\cdot 36/\binom{52}{2}$	$\binom{12}{2}/\binom{52}{2}$
1	$3\cdot 36/\binom{52}{2}$	$\frac{36+3\cdot 12}{\binom{52}{2}}$	$1\cdot 12/\binom{52}{2}$
2	$\binom{3}{2}/\binom{52}{2}$	$3/\binom{52}{2}$	0

Good Quiz Practice

Check these!!

$P(X=0) = p(0,0) + p(0,1) + p(0,2)$

↳ sum of first column

$P(Y=1) = p(0,1) + p(1,1) + p(2,1)$

↳ sum of middle row.