Lecture 17

Important Examples of continuous random variables:

Uniform Distribution:

A continuous r.v. X is said to be uniformly distributed on the finite interval $[a,b]$ $X \sim Unif(a,b)$ if its pdf is:

$$p(x) = \frac{1}{b-a} \times \begin{cases} 1, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$$

- $I = [c,d] \subset [a,b]$ $\quad P(X \in I) = ?$

$$P(c \leq X \leq d) = \frac{1}{b-a} \int_c^d dx = \frac{d-c}{b-a}$$

Note: This shows the probability only depends on the length of the segment, not where the segment is.

Rmk: $X \sim Unif(a,b)$, then X is s-integrable $\forall\, s \geq 1$.

$$\mu_k[X] = \int_a^b \frac{1}{b-a} x^k dx = \frac{1}{b-a} \int_a^b x^k dx$$

$$= \frac{b^{k+1} - a^{k+1}}{(b-a)(k+1)}$$

$$\mathbb{E}[X] = \mu_1[X] = \frac{b^2 - a^2}{(b-a)2} = \frac{b+a}{2}$$

$$\mu_2[X] = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

Exponential Distribution: A continuous r.v. $T \sim Exp(\lambda)$ if its pdf

$$p(t) = \begin{cases} 0, & t < 0 \\ \lambda e^{-t\lambda}, & t \geq 0 \end{cases}$$

λ is called the rate, $\lambda > 0$

$$\int_{-\infty}^{\infty} p(t)\,dt = \int_0^{\infty} \lambda e^{-t\lambda}\,dt = \lambda\left(-\frac{1}{\lambda}e^{-t\lambda}\right)\Big|_0^{\infty} = 1$$

cdf: $F(x) = P(X = x) = \int_{-\infty}^{x} p(t)\,dt = \int_0^x p(t)\,dt$

$$= \int_0^x \lambda e^{-t\lambda}\,dt = \lambda\left(-\frac{1}{\lambda}e^{-t\lambda}\right)\Big|_0^{x} = 1 - e^{-x\lambda}$$

- The function $G(t) = 1 - F(t) = P(T > t)$ is called the survival function.

  ↳ Note that $P(T > t) = \lambda \int_t^{\infty} e^{-\lambda s}\,ds = e^{-\lambda t}$

- Memoryless property:

$$P(T > t_0 + t \mid T > t) = P(T > t_0) \quad \forall\, t_0, t > 0$$

Proof:

$$P(T > t_0 + t \mid T > t) = \frac{P(T > t_0 + t) \cap (T > t)}{P(T > t)} = \frac{P(T > t_0 + t)}{P(T > t)} = \frac{e^{-\lambda(t_0 + t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda t_0}$$

$$= P(T > t_0)$$

$\boxtimes$

Rmk: $T \sim Exp(\lambda)$, then again t is s-integrable $\forall s \geq 1$

$$\mu_k[X] = \int_{\mathbb{R}} t^k p(t)\,dt = \int_0^{\infty} t^k \lambda e^{-\lambda t}\,dt$$

$$= \lambda \int_0^{\infty} t^k e^{-\lambda t}\,dt = \left[x = \lambda t,\ t = \frac{x}{\lambda},\ t^k = \lambda^{-k} x^k,\ dt = \frac{1}{\lambda}dx\right]$$

$$= \lambda \int_0^{+\infty} \lambda^{-k} x^k e^{-x} \frac{1}{\lambda}\,dx = \lambda^{-k} \int_0^{\infty} x^k e^{-x}\,dx = \lambda^{-k} k!$$

$$= \int_0^{\infty} x^k e^{-x}\,dx = \Gamma(k+1) = k!$$

$$\mathbb{E}[X] = \mu_1 = \frac{1}{\lambda}$$

$$\mu_2 = \frac{2}{\lambda^2}$$

$$var[X] = \mu_2 - (\mu_1)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

- Assume we have a Bernoulli experiment ever δ seconds with probability p_δ.

- Let T be the time we need to wait until the 1st success,
  $T \sim geom(p_\delta)$

- Let $T = n\delta$ then

$$G_\delta(t) = P(T > t) = (1 - p_\delta)^n$$

$$= (1 - p_\delta)^{\frac{t}{\delta}} = \left[(1-p_\delta)^{-\frac{1}{p_\delta}}\right]^{-p_\delta \frac{t}{\delta}}$$

- Assume $\rho_0 = \delta\lambda$

$$G_\delta(t) = \Big[\underbrace{(1-\delta\lambda)^{-\frac{1}{\delta\lambda}}}_{\xrightarrow[\delta\to 0]{} e} \Big] \xrightarrow[\delta \to 0]{} e^{-\lambda t}$$

- **The normal (Gaussian) distribution:**

$X \sim N(\mu, \sigma^2)$ if its pdf is given by

$$d_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}$$

Gamma and Beta Functions

Def: The gamma function is the function $\Gamma : (0,\infty) \to \mathbb{R}$ defined by:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

The beta function of two positive variables defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad x, y > 0$$

Prop: The following properties hold:

i) $\Gamma(1) = 1$

ii) $\Gamma(x+1) = x\Gamma(x)$, $\forall\, x > 0$

iii) For any integer $n = 1, 2, \dots$, we have : $\Gamma(n) = (n-1)!$

iv) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

v) Euler's formula : $\int_0^1 s^{x-1}(1-s)^{y-1} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y) \quad \forall\, x, y > 0$

Special Case: Let $X \sim N(0,1)$

X is standard normal / standard gaussian

- The cdf of a standard normal variable is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \, dt$$

↳ there are good approximation charts for these.

Gamma Distributions

$X \sim \text{Gamma}(v, \lambda)$ if its pdf:

$$g_v(x, \lambda) = \begin{cases} \frac{\lambda^v}{\Gamma(v)} x^{v-1} e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Rmk: $\text{Gamma}(1, \lambda) = \text{Exp}(\lambda)$