# Lecture 16

## • Continuous Random Variables

**Def:** A random variable $X$ is called a continuous r.v. if $\exists$ a function $p : \mathbb{R} \to [0, \infty)$ s.t.

$$P(X \le x) = \int_{-\infty}^{\infty} \boxed{p(s)}\, ds, \quad \forall x \in \mathbb{R}$$

probability density function (pdf) of $X$

• The random variable is said to be concentrated on an interval $I$ if $p(x) = 0$, $\forall x \notin I$

• Rmk: a) "$P(X \in [x, x+dx]) \approx p(x)dx$"

The cdf of $X$ is $F(x) = P(X \le x)$

$$= \int_{-\infty}^{x} p(s)\,ds$$

• In particular, $1 = P(X < \infty) = \int_{-\infty}^{\infty} p(s)\,ds$

• Moreover $F'(x) = p(x)$

b) $c \in \mathbb{R}$

$P(X < c) = P(X \leq c) = F(c)$

Indeed

$$P(X < c) = \lim_{\varepsilon \to 0} P(X \leq c - \varepsilon) = \lim_{\varepsilon \to 0} \int_{-\infty}^{c-\varepsilon} p(s)\,ds = \int_{-\infty}^{c} p(s)\,ds = P(X \leq c)$$

∎

c) $\forall\, a, b \in \mathbb{R}$ s.t. $a \leq b$

$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a)$$
$$= F_X(b) - F_X(a) = \int_a^b p(s)\,ds$$

d) $\forall\, c \in \mathbb{R}$

$P(X = c) = 0$

$P(x = c) = P(c \leq X \leq c) = F_X(c) - F_X(c) = 0$

• Prop: Any function $p : \mathbb{R} \to [0, \infty)$ s.t. $\int_{-\infty}^{\infty} p(s)\,ds = 1$ is the pdf of some continuous r.v.

• Def: $X$ is a continuous r.v. with pdf $p$, $s \in \mathbb{R}$, $s \geq 1$

a) We say that $X$ is s-integrable if $\int_{-\infty}^{\infty} |x|^s p(x)\,dx < \infty$ $(X \in L^s)$

We say that $X$ is integrable if it is 1 integrable.

—— // —— $X$ is square integrable

—— // —— 2-integrable

b) If $X$ is integrable then $\mathbb{E}[X] = \int_{\mathbb{R}} x p(x)\, dx$

c) If for $n = 1, 2, 3,$ $X$ is $n$-integrable then the $n^{th}$ moment of $X$

$$\mu_n[X] = \int_{\mathbb{R}} x^n p(x)\, dx \quad , \quad \mu_1[X] = \mathbb{E}[X]$$

d) If $X \in L^2$ and $\mu = \mathbb{E}[X]$, then $var[X] = \int_{\mathbb{R}} (x-\mu)^2 p(x)\, dx$

$$\sigma[X] = \sqrt{var[X]}$$

<u>Rmk</u>: If $X \in L^s$ for some $s \geq 1$ then $X \in L^r$ $\forall r \in [1, s]$

<u>Prop</u>: If $X \in L^2$, then $var[X] = \mu_2[X] - (\mu_1[X])^2$

<u>Proof</u>:

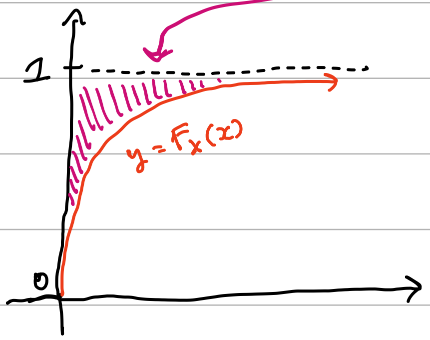
$$Var[X] = \int_{\mathbb{R}} (x-\mu)^2 p(x)\, dx = \int_{\mathbb{R}} (x^2 - 2\mu x + \mu^2) p(x)\, dx$$

$$= \underbrace{\int_{\mathbb{R}} x^2 p(x)\, dx}_{\mu_2[X]} - 2\mu \underbrace{\int_{\mathbb{R}} x p(x)\, dx}_{\mu} + \mu^2 \underbrace{\int_{\mathbb{R}} p(x)\, dx}_{1}$$

$$= \mu_2[X] - 2\mu^2 + \mu^2 = \mu_2 - (\mu_1)^2 \quad \blacksquare$$

**Prop:** Suppose that $X$ is non-negative continuous r.v. then $\mathbb{E}[X] = \int_0^\infty P(X > x)\,dx$

**Rmk:** $X \geq 0$, so $\mathbb{E}[X]$ is this area




**Thm:** (The Law of Subconsious statisian)

$X$ - cont. r.v. with pdf $p(x)$. Let $f(x)$ be a function s.t. $f(x)$ is either a continuous r.v. or a discrete r.v. If $f(x) \in L^1$, then

$$\mathbb{E}[f(x)] = \int_{\mathbb{R}} f(x)p(x)\,dx$$

**Cor:** $X$ - continuous r.v., with $\mathbb{E}[X] = \mu$, $X \in L^1$,

Then $\forall\, a, b \in \mathbb{R}$

$$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$$

if $X \in L^k$ then

$$\mu_k[X] = \mathbb{E}[X^k]$$

$$\mu_k[cX] = c^k \mu_k[X] \quad \forall\, c \in \mathbb{R}$$

If $X \in L^2$, then:

$$\mathrm{var}[X] = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\mathrm{var}[aX+b] = a^2\, \mathrm{var}[X] \quad \forall a, b \in \mathbb{R}$$

Prop: Monotonicity of $\mathbb{E}$

$X$ - continuous r.v., $f, g : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) \leq g(x)$ $\forall x \in \mathbb{R}$

Then:

$$\mathbb{E}[f(X)] \leq \mathbb{E}[g(X)]$$

Prop: Markov Inequality

- Let $X$ be an integrable, continuous, non-negative r.v.

- Then $\forall c > 0$

$$P(X > c) \leq \frac{1}{c} \mathbb{E}[X]$$

Proof: $X \geq 0 \Rightarrow p(x) = 0 \quad \forall x < 0$

$$\mathbb{E}[X] = \int_0^{+\infty} x p(x)\, dx$$

$$cP(X > c) = \int_c^{+\infty} c p(x)\, dx \leq \int_c^{+\infty} x p(x)\, dx \leq \int_0^{+\infty} x p(x)\, dx = \mathbb{E}[X] \quad \blacksquare$$

## Thm: Chebychev's Inequality

$X \in L^2$, $\mu = \mathbb{E}[X]$, $\sigma = \sqrt{\mathrm{var}[X]}$. Then $\forall\, c, r > 0$

$$P(|X-\mu| > c\sigma) \le \frac{1}{c^2}$$

take $r = c\sigma$ to prove this inequality

$$P(|X-\mu| > r) \le \frac{\sigma^2}{r^2}$$

(second inequality)

__Proof__: Consider $Y = (X-\mu)^2$. Note that $|X-\mu| > r \Rightarrow Y > r^2$

Markov's inequality $\Rightarrow \forall c > 0$

$$P(Y > r^2) \le \frac{1}{r^2}\mathbb{E}[Y] = \frac{1}{r^2}\underbrace{\mathbb{E}[(X-\mu)^2]}_{\mathrm{Var}[X]} = \frac{1}{r^2}\sigma^2$$