Lecture 15

Warm Up: A factory produces light bulbs, and each bulb is defective with probability 0.2. A quality inspector randomely selects 10 bulbs. Find the prob that 7 of selected bulbs are defective. What is the expected number of defective balls?

1) X = # of defective balls
2) $X \sim \text{Binomial}(10, 0.2)$
3) $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$
   $\hookrightarrow P(X=7) = \binom{10}{7}(0.2)^7(0.8)^3$

4) $\mathbb{E}[X] = np$
   $\hookrightarrow \mathbb{E}[X] = 10(0.2) = 2$

Functions of Discrete Random Variables

$X \sim$ discrete random variable with range $\mathcal{X}$ and pmf $p : \mathcal{X} \to [0,1]$

Then for any $f : \mathbb{R} \to \mathbb{R}$, we get a new random variable $f(X)$ such that the following holds:

- if the value of X is $x \in \mathcal{X}$, then the value of $f(X)$ is $f(x)$
- $P(F(X)=y) = \sum_{\substack{x \in \mathcal{X} \\ f(x)=y}} p(x)$

$\boxed{\text{Ex}}$ Suppose X is discrete random variable with

$$\mathcal{X} = \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$$

and pmf $p_n = P(X=n)$ $\forall n \in \mathbb{Z}$ then X^2 is a random variable with range $\{0, 1^2, 2^2, \dots\} = \{0, 1, 4, 9, \dots\}$

$$P(X^2=0) = p_0$$
$$P(X^2=1) = p_{-1} + p_1$$
$$P(X^2=49) = p_{-7} + p_7$$
$$P(X^2=2) = 0$$

Thm: Law of Subconsious Statitician

$X \sim$ discrete random variable with $\mathcal{X} = \{x_1, x_2, \dots\} \subset \mathbb{R}$ and pmf $p(x_k) = \mathbb{P}(X=x_k)$ $\forall k \geq 1$

Then for any function $f: \mathbb{R} \to \mathbb{R}$, if the expectation of $f(X)$ exists, it is given by $\mathbb{E}[f(X)] = \sum_{k \geq 1} f(x_k)\, p(x_k)$

Cor: Linearity of the Expectation

Given $X \sim$ Discrete random variable and f, g, be two functions. Then:

$$\mathbb{E}[f(X) + g(X)] = \mathbb{E}[f(X)] + \mathbb{E}[g(X)]$$

In particular for any a, b we have $\mathbb{E}[aX+b] = a(\mathbb{E}[X]) + b$

Proof:

range of X

$p_k = P(X = x_k)$, $\mathcal{X} = \{x_1, \dots\}$

$$\mathbb{E}[f(x) + g(x)] = \sum_{k\geq 1} p_k [f(x_k) + g(x_k)]$$

$$= \underbrace{\sum_{k\geq 1} p_k f(x_k)}_{\mathbb{E}[f(x)]} + \underbrace{\sum_{k\geq 1} p_k g(x_k)}_{\mathbb{E}[g(x)]}$$

$$\mathbb{E}[aX+b] = \sum_{k\geq 1} p_k (ax_k + b) = \underset{a}{a} \underbrace{\sum p_k x_k}_{\mathbb{E}[X]} + \underset{b}{b} \underbrace{\sum p_k}_{1}$$

Recall: $\mathrm{var}[X] = \sum_{x \in \mathcal{X}} (x-\mu)^2 p(x)$

$\mu = \mathbb{E}[X]$

Cor: Let X be 2-integrable discrete random variable with the mean μ.

Then $\mu_n[X] = \mathbb{E}[X^n] \quad \forall n \in \{1, 2, \dots\}$

$\mathrm{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[(X-\mu)^2]$

Moreover, for any $a, b \in \mathbb{R}$ we have $\mathrm{var}[aX+b] = a^2 \mathrm{var}[X]$

Proof: $\mu_X = \mathbb{E}[X]$. We set $Y = aX + b$ then $\mu_Y = \mathbb{E}[Y] = a\mu_X + b$. Hence

$Y - \mu_Y = (aX+b) - (a\mu_X + b) = a(X - \mu_X)$

$\mathrm{var}[Y] = \mathbb{E}[(Y-\mu_Y)^2] = \mathbb{E}[a^2 (X-\mu)^2]$

$= a^2 \mathbb{E}[(X-\mu_X)^2]$

$= a^2 \mathrm{var}[X]$

Cor: $X \sim$ discrete random variable, $\mathcal{X} \subset \{0, 1, 2, \dots\}$

Denote by G_X its pgf. The $\forall\, s \in [0, 1]$

$$G_X(s) = \mathbb{E}[s^X]$$

Proof: $p_n = p(X=n)$

$$\underbrace{\mathbb{E}[s^X] = p_0 s^0 + p_1 s^1 + p_2 s^2}_{\text{by Thm}} + \dots \overset{\text{by definition}}{=} G_X(s)$$

Cor: Monotonicity of the Expectation

X, $f, g : \mathcal{X} \to \mathbb{R}$ such that $f(x) \leq g(x)$ $\forall x \in \mathcal{X}$

Then $\mathbb{E}[f(X)] \leq \mathbb{E}[g(X)]$

Proof:

$$\mathbb{E}[f(X)] = \sum_{x \in \mathcal{X}} p(x) f(x) \leq \sum_{x \in \mathcal{X}} p(x) g(x) = \mathbb{E}[g(X)]$$

Prop: Markov Inequality

$X \sim$ discrete random variable, $\mathcal{X}$ consisting of only $\boxed{\text{non-negative}}$ numbers and some pmf : $\mathcal{X} \to [0,1]$. Then:

$$P(X > c) \leq \frac{1}{c} \mathbb{E}[X] \quad \forall c > 0$$

Proof: $X \geq 0$, $c > 0$

$$P(X > c) = \sum_{\substack{x > c \\ x \in \mathcal{X}}} P(X = x)$$

because range is nonnegative

$$cP(X > c) = \sum_{\substack{x > c \\ x \in \mathcal{X}}} c\,p(x) < \sum_{\substack{x > c \\ x \in \mathcal{X}}} x\,p(x) \boxed{\leq} \sum_{x \in \mathcal{X}} x\,p(x) = \mathbb{E}[X]$$

Thm: Chebyshev's inequality

$X \sim$ discrete random variable, $X \in L^2$, $\mu = \mathbb{E}[X]$, $\sigma = \sqrt{\text{var}[X]}$

Then for any $c, r > 0$

$$\boxed{P(|X - \mu| > c\sigma) \leq \frac{1}{c^2}}$$

$$P(|X - \mu| > r) \leq \frac{\sigma^2}{r^2}$$

This inequality tells us how much the value of the random variable deviates from the mean.

Proof :

$c^2\sigma^2 P(|X-\mu| > c\sigma)$

$$= c^2\sigma^2 \sum_{\substack{x\in\mathcal{X}\\|x-\mu|>c\sigma}} p(x) = \sum_{\substack{x\in\mathcal{X}\\|x-\mu|>c\sigma}} c^2\sigma^2 p(x) \leq \sum_{\substack{x\in\mathcal{X}\\|x-\mu|>c\sigma}} (x-\mu)^2 p(x) \leq \sum_{x\in\mathcal{X}} (x-\mu)^2 p(x) = \mathrm{var}[X] = \sigma^2$$