# Lecture 14

- <u>Prop</u>: Suppose that $X$ is an integrable discrete random variable with range contained in $0, 1, 2, \ldots$, then we have $\mathbb{E}[X] = \sum_{n \geq 0} P(X > n)$

- $\boxed{\text{Ex}}$ Let $X$ be the number of rolls a die until the first 6 appears. $X \sim \text{Geom}\left(\frac{1}{6}\right)$. We know $P(X > n) = \left(\frac{5}{6}\right)^n$.

  Hence:

$$\mathbb{E}[X] = 1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \ldots + \left(\frac{5}{6}\right)^n + \ldots = \frac{1}{1 - \frac{5}{6}} = 6.$$

<u>Rmks</u>:

- The <u>mean</u> shows the "center"
- The <u>variance</u> tells how far values typically lie from that center (spread)
- The <u>standard deviation</u> is the typical distance from the mean.

<u>Question</u>: Why not just $\mathbb{E}[X - \mu]$? Why $\mathbb{E}[(X - \mu)^2]$?

↳ <u>Answer</u>: Because $\mathbb{E}[X - \mu] = 0$ always.

# Probability Generating Functions:

- **Def**: (The probability generating function). Let $X$ be a discrete random variable with range $N_0 = \{0, 1, 2, \dots\}$. For $n \in N_0$ we set $p_n = \mathbb{P}(X = n)$.

  The probability generating function (pgf) of $X$ is the function

$$G_X : [0,1] \to \mathbb{R}, \quad G_X(s) = p_0 + p_1 s + \dots + p_n s^n + \dots = \sum_{n=0}^{\infty} p_n s^n$$

- $\boxed{\text{Ex}}$ Let $X$ be a random variable with range $\{0,1,2,3\}$ and pmf,

  $p(0) = p(3) = \frac{1}{8}$, $p(1) = p(2) = \frac{3}{8}$

$$G_X(s) = \frac{1}{8} + \frac{3}{8}s + \frac{3}{8}s^2 + \frac{1}{8}s^3$$

- Any statistical quantity associated to a random variable can be expressed in terms of its probability generation function.

**Prop**: Suppose that $X$ is a discrete random variable with range contained in $\mathbb{N}_0 = \{0,1,2,\dots\}$

If $G_X(s)$ is the pgf of $X$ then:

$$G_X(1) = 1$$

$$\mathbb{E}[X] = G_X'(1)$$

$$M_2[X] = G_X''(1) + G_X'(1)$$

$$\text{var}[X] = G''_X(1) + G_X'(1) - [G'_X(1)]^2$$

$X \sim Ber(p)$ $\quad \mathcal{X} = \{0,1\}$

$$G_X(s) = p_0 + p_1 s = (1-p) + ps$$

$G'_X(s) = p \qquad \mathbb{E}[X] = p$

$G''_X(s) = 0 \qquad \text{var}[X] = 0 + p - p^2$

$\qquad\qquad\qquad\qquad = p(1-p)$

$X \sim \mathrm{Bin}(n,p)$ $\quad \mathcal{X} = \{0,1,2,\dots,n\}$

$$G_X(s) = p_0 + p_1 s + p_2 s^2 + \dots + p_n s^n$$
$$= \sum_{k=0}^{n} P(X=k) s^k$$

$$= \binom{n}{0}(1-p)^n + \binom{n}{1}(1-p)^{n-1} p s + \binom{n}{2}(1-p)^{n-2} p^2 s^2 + \dots + \binom{n}{n} p^n s^n$$
$$= \left[(1-p) + ps\right]^n \quad \text{by Newton's binomial formula}$$
$$= \left[G_{\mathrm{Ber}(p)}(s)\right]^n$$

$$G_X'(s) = np\left[(1-p)+ps\right]^{n-1} \qquad G_X''(s) = n(n-1)p^2\left[(1-p)+ps\right]^{n-2}$$
$$G_X'(1) = np = \mathbb{E}[X] \qquad G_X''(1) = n(n-1)p^2$$

$$\mathrm{var}[X] = n(n-1)p^2 + np - n^2p^2$$
$$= np\left[(n-1)p + 1 - np\right) = np(1-p)$$

$X \sim \mathrm{Geom}(p)$

$$G_X(s) = \overset{=0}{p_0} + p_1 s + \dots = ps + (1-p)ps^2 + (1-p)^2 p s^3 + \dots$$
$$= ps\left[1 + (1-p)s + (1-p)^2 s^2 + \dots\right] = \frac{ps}{1-(1-p)s}$$

$$G_X'(s) = \frac{p\left[1-(1-p)s\right] + ps(1-p)}{\left[1-(1-p)s\right]^2} = \frac{p}{\left[1-(1-p)s\right]^2}$$

recall $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

$$G_X'(1) = \frac{p}{\left[1-(1-p)\right]^2} = \frac{1}{p} = \mathbb{E}[X]$$

$$G_X''(s) = \dots$$

$$\mathrm{var}[X] = \dots$$

## $X \sim NegBin(r,p)$

$$G_X(s) = \sum_{n=r}^{\infty} P(X=n)s^n$$

because $P(X=n)$ for $n<r$ is always 0.

$$= \sum_{n=r}^{\infty} \binom{n-1}{r-1} p^n (1-p)^{n-r} s^n$$

$$= p^r (1-p)^{-r} \sum_{n=r}^{\infty} \binom{n-1}{r-1} (1-p)^n s^n$$

$n = r+k$

$$= p^r (1-p)^{-r} \sum_{k=0}^{\infty} \binom{r+k-1}{r-1} (1-p)^{r+k} (s)^{r+k}$$

$$= p^r s^r \underbrace{\sum_{k=0}^{\infty} \binom{r+k-1}{r-1} (1-p)^k s^k}_{[1-(1-p)s]^{-r}}$$

$$G_X(s) = p^n s^n \left[1-(1-p)s\right]^{-r}$$

$$= \left(\frac{ps}{1-[1-p]s}\right)^r$$

## $X \sim Poi(\lambda)$

$$G_X(s) = \sum_{n=0}^{\infty} p_n s^n = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} s^n$$

$$= e^{-\lambda} + \frac{\lambda}{1!} e^{-\lambda} s + \frac{\lambda^2}{2!} e^{-\lambda} s^2 + \dots$$

$$= e^{-\lambda}\left(1 + \frac{\lambda s}{1!} + \frac{(\lambda s)^2}{2!} + \dots\right) = e^{-\lambda} \cdot e^{\lambda s} = e^{\lambda(s-1)}$$

$G'_X(s) = \lambda e^{\lambda(s-1)}$ $\qquad$ $G''_X(s) = \lambda^2 e^{\lambda(s-1)}$

$G'_X(1) = \lambda$ $\qquad$ $G''_X(1) = \lambda^2$

$\mathbb{E}[X] = \lambda$ $\qquad$ $var[X] = \lambda$