# Lecture 13

<u>Claim</u>: For a fixed $\lambda$ and $n$ large enough $B(n,p) = Poi(\lambda)$ with $p = \frac{\lambda}{n}$.

<u>Proof</u>:

$$P(X=K) = \binom{n}{K} p^K (1-p)^{n-K}$$

$$= \frac{n(n-1)\cdots(n-K+1)}{K!} \frac{\lambda^K}{n^K} \left(1 - \frac{\lambda}{n}\right)^{n-K}$$

$$= \underbrace{\frac{n(n-1)\cdots(n-K+1)}{n^K}}_{\lim_{n\to\infty} = 1} \frac{\lambda^K}{K!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{n-K}}_{\lim_{n\to\infty} = e^{-\lambda}}$$

$$= \frac{\lambda^K}{K!} e^{-\lambda} \quad \blacksquare$$

## Statistical invariance of discrete random variables:

**Def**: Let $X$ be a discrete random variable with range $\mathcal{X} = \{x_1, x_2, \dots\}$ and pmf $p: \mathcal{X} \to [0,1]$

a) $s \in [1, \infty)$, We say that $X$ is s-integrable if

$$\sum_{x \in \mathcal{X}} |x|^s p(x) < \infty$$

b) if $X$ is integrable then the expectation or mean is the real number

$$E[X] = \sum_{x \in \mathcal{X}} x p(x)$$

$$\text{"} \; x_1 P(X = x_1) + x_2 P(X = x_2) + \dots \; \text{"}$$

c) If $n \in \{1, 2, 3, \dots\}$ and $X$ is n-integrable, then the $n^{th}$ moment of $X$.

$$\mu_n[X] = \sum_{x \in \mathcal{X}} x^n p(x) \quad \underset{=}{\text{Rmk}} \quad \mathbb{E}[X^n]$$

**Rmk** : If a discrete random variable is s-integrable for some $s \geq 1$, then it is r-integrable for any $r \in [1, s]$

$\boxed{\text{Ex}}$: Casino

$P(\text{win}) = 0.4929$

$P(\text{lose}) = 1 - 0.4929 = 0.5071$

- Assume each time gambler looses the casino gains \$1
  ↳ and if gambler wins the casino looses \$1

casino gains \$1

$$p = P(X = 1) = 0.5071$$

$$1 - p = P(X = -1) = 0.4929$$

casino looses \$1

$$\mathbb{E}[X] = 1 \cdot P(X=1) + (-1) P(X = -1)$$

$$= p - (1-p) = 2p - 1 = 0.0142$$

**Def** : $X$ - discrete random variable with range $\mathcal{X}$ and pmf $p$.

If $X \in L^2$ and its mean $\mu = \mu_1[X]$, then the variance of $X$:

$$\text{var}[X] = \sum_{x \in \mathcal{X}} (x - \mu)^2 p(x) = \mathbb{E}[(X - \mu)^2]$$

**Def**: The standard deviation :

$$\sigma[X] = \sqrt{\text{var}[X]}$$

Rmk: $X \in L^2 \implies var[X] < \infty$, even when the range is unbounded

Prop: $X$ discrete random variable with $\mathcal{X}, p$. If $X \in L^2$, then

$$var[X] = M_2[X] - (M_1[X])^2$$
$$\left( = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \right)$$

$$var[cX] = c^2 var[X] \quad \forall c \in \mathbb{R} / \{0\}$$

$$\mathbb{E}[cX] = \sum_{x \in \mathcal{X}} cx\, p(cx)$$

$$P(cX = cx) = P(X = x)$$

proof: $\mu = \mathbb{E}[X]$

$$var[X] = \sum_{x \in \mathcal{X}} (x-\mu)^2 p(x)$$

$$= \sum_{x \in \mathcal{X}} (x^2 - 2\mu x + \mu^2)\, p(x)$$

$$= \underbrace{\left( \sum_{x \in \mathcal{X}} x^2 p(x) \right)}_{\mathbb{E}[x^2]} - 2\mu \underbrace{\sum_{x \in \mathcal{X}} x\, p(x)}_{\mathbb{E}[x]} + \left( \mu^2 \underbrace{\sum_{x \in \mathcal{X}} p(x)}_{1} \right)$$

$$= \mathbb{E}[x^2] - 2(\mathbb{E}[x])^2 + (\mathbb{E}[x])^2$$

$$= M_2[X] - (M_1[X])^2$$

$$\mathrm{var}[cX] = \mu_2[cX] - (\mu_1[cX])^2$$
$$= c^2 \mu_2[X] - (c\mu_1[X])^2$$
$$= c^2\left(\mu_2[X] - (\mu_1[X])^2\right) = c^2 \,\mathrm{var}[X]$$

$$\mu_n[cX] = c^n \mu_n[X]$$

$\boxed{\text{Ex}}$ Let $X$ be a discrete random variable with $\mathcal{X} = \{0,1,2,3\}$ and pmf $p$: $p(0) = p(3) = \frac{1}{8}$, $p(1) = p(2) = \frac{3}{8}$

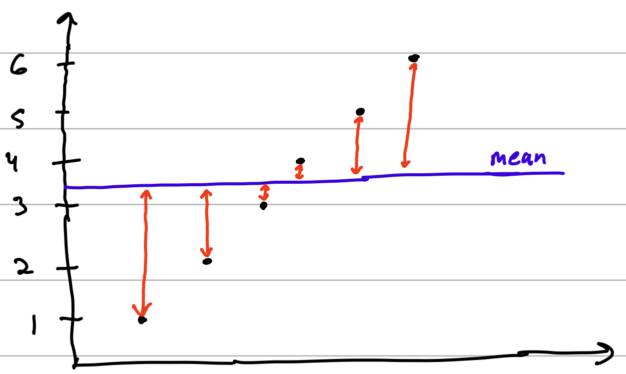
$$\mathbb{E}[X] = 0p(0) + 1p(1) + 2p(2) + 3p(3)$$
$$= 0\left(\tfrac{1}{8}\right) + 1\left(\tfrac{3}{8}\right) + 2\left(\tfrac{3}{8}\right) + 3\left(\tfrac{1}{8}\right) = \frac{3}{2}$$

"In average number of girls in a family with 3 kids is 1.5 girls"

$$\mathrm{var}[X] = 0^2 p(0) + (1)^2 p(1) + (2)^2 p(2) + (3)^2 p(3) - \left(\tfrac{3}{2}\right)^2 = \frac{3}{4}$$

$\boxed{\text{Ex}}$ Roll a fair die $\mathcal{X} = \{1,2,3,4,5,6\}$

$$\mathbb{E}[X] = \tfrac{1}{6}(1) + \tfrac{1}{6}(2) + \tfrac{1}{6}(3) + \tfrac{1}{6}(4) + \tfrac{1}{6}(5) + \tfrac{1}{6}(6) = 3.5$$
$$\mathrm{var}[X] = \tfrac{1}{6}(1^2 + 2^2 + \ldots + 6^2) - (3.5)^2 = \frac{35}{12} \approx 2.916$$




variance is sum of squared residuals.

$\boxed{\text{Ex}}$ Roll a pair of dice, let $S$ be the sum

$\mathcal{X} = \{2, 3, \dots, 12\}$

$\mathbb{E}[S] = 2P(S=2) + 3(S=3) + \dots + 12P(S=12)$

$$\mathbb{E}[S] = \frac{(2\cdot 1) + (3\cdot 2) + \dots + (7\cdot 6) + (8\cdot 5) + \dots + 12(1)}{36} = \frac{252}{36} = 7$$