Lecture 12

• Negative Binomial distribution $\quad X \sim NegBin(r,p)$

• Fix a natural number r. Repeat independent $Ber(p)$ trials until we register r successes.

Let X be the number of trials required. Clearly $P(X<r)=0$

$$P(X=k) = \underbrace{p}_{\text{probability of success}} \binom{k-1}{r-1} p^{r-1} (1-p)^{\overbrace{k-r}^{(k-1)-(r-1)}}$$

Rmk : Note that

1) $Geom(p) = NegBin(1,p)$

2) if $X_1, X_2, \ldots, X_k$ are independent geometric random variables, with the same prob. of success p, then :

$$X_1 + X_2 + \ldots + X_k \sim NegBin(k,p)$$

Ex Banach's problem
A kid has n candies in each pocket From time to
reaches a hand into tia mrandomly e selected pocket to
take one candy What is the probability that when he
First discovers a pocket to be empty the other pole contains exactly m candies

Sol : L_m = "the first empty pocket is the left one and there are m candies in the right pocket"

R_m : "Same but roles of pockets reversed"

Finding $P(R_m \cup L_m) = P(L_m) + P(R_m)$ since $R_m \cap L_m = \emptyset$
$= 2P(L_m)$

Success : "reaching the left pocket", $p = \frac{1}{2}$ (Since kid chooses randomely left or right)

L_m : $(n+1)+(n-m)$ trails during which $(n+1)$ successes, with $(n+1)^{\text{th}}$ occuring at the last trail.

$X \sim NegBin(n+1, \frac{1}{2})$

$$P(L_m) = P[X = \underbrace{n+1+(n-m)}_{2n+1-m}] = \binom{2n+1-m-1}{n+1-1} \frac{1}{2^{2n+1-m}} = \frac{\binom{2n-m}{n}}{2^{2n+1-m}}$$

$$\text{So} \quad 2P(L_m) = \frac{\binom{2n-m}{n}}{2^{2n-m}} = P(R_m \cup L_m)$$

Ex A biased coin has probability $p = 0.4$ of heads. The coin is tossed until 3 heads are observed. Let X be the number of tosses required.

a) Identify distribution and parameters of X!

$$X \sim \text{NegBin}(\overset{r}{3}, \overset{p}{0.4})$$

b) Write the pmf of X:

$$P(X=k) = \binom{k-1}{2}(0.4)^3(0.6)^{k-3} \quad \text{for } k \geq 3$$

c) Compute $P(X=3)$ and $P(X=5)$

$$P(X=3) = \binom{3-1}{2}(0.4)^3(0.6)^{3-3} = (0.4)^3$$

$$P(X=5) = \binom{5-1}{2}(0.4)^3(0.6)^{5-3} = \binom{4}{2}(0.4)^3(0.6)^2$$

d) What is the smallest possible value of X?

↳ trivially 3

- Hypergeometric Distribution $X \sim HGeom(b, y, n)$

  - Assume you have a bin with b blue balls and y yellow balls.
    ↳ So $b+y$ is total # of balls.

  - Select n balls at random from the bin. Let X be the number of blue balls among the selected ones.

  - The range of X is not necessarily 0 to n, it depends on the distribution of balls in your bin.

    ↳ So range of X is:

$$\max(0, n-y) \le X \le \min(b, n)$$

  - pmf: $P(X=k) = \dfrac{\text{"good"}}{\text{"all"}} = \dfrac{\binom{b}{k}\binom{y}{n-k}}{\binom{b+y}{n}}$

$\boxed{\text{Ex}}$ (Number of aces in a random poker hand) A 5 card hand is dealt at random without replacement. Let X be the number of aces in the hand.

a) Identify the distribution of X and its parameters

$$X \sim HGeom(4, 48, 5)$$

b) Find the range of X:

$$\mathcal{X} = \{0, 1, 2, 3, 4\}$$

c) Write the pmf of X:

$$P(X=k) = \frac{\binom{4}{k}\binom{48}{5-k}}{\binom{52}{5}}, \quad k \in \mathcal{X}$$

- Poisson Random Variable $\quad X \sim Poi(\lambda)$

- A Poisson random variable with parameter $\lambda > 0$, is a discrete random variable with range $\mathcal{X} = \{0, 1, 2, \dots\}$ and pmf

$$P(X=k) = e^{-\lambda}\left(\frac{\lambda^k}{k!}\right) \quad \forall\ k = 0, 1, 2, \dots$$

- The Poisson distribution typically models the occurance of rare events in a given unit of time. Then $p_k(\lambda)$ would be the probability that k of these events took place during that unit of time.

$\boxed{\text{Ex}}$ Calls arrive at help desk according to Poisson distribution with a mean rate $\lambda = 3$ per hour. Let X be the number of calls recieved in one hour.

a) Write the pmf of X

$$P(X=k) = e^{-3}\,\frac{3^k}{k!}$$

b) Compute $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

c) Find the mode of X

↳ value of X with highest probability

↳ look at proposition below

• **Proposition** : The mode of a Poisson distribution with parameter λ is $\lfloor \lambda \rfloor$ if λ is NOT an integer and both λ and $(\lambda-1)$ if λ is an integer.

Proof : $P(X=n) = \dfrac{e^{-\lambda}\lambda^n}{n!}$ $\quad n = 0, 1, 2, \dots$

$$\frac{P(X=n)}{P[X=(n-1)]} = \frac{e^{-\lambda}\dfrac{\lambda^n}{n!}}{e^{-\lambda}\dfrac{\lambda^{n-1}}{(n-1)!}} = \frac{\lambda}{n}$$

If $n < \lambda \rightarrow P(X=n) > P(X=n-1)$

If $n > \lambda \rightarrow P(X=n) < P(X=n-1)$