Lecture 11

<u>Proposition</u>: Suppose that $\mathcal{X} \subset \mathbb{R}$ is finite or countable set. $\mathcal{X} = \{x_1, x_2, \dots\}$. A function $p: \mathcal{X} \to [0,1]$ is the probability distribution of a discrete random variable with range $\mathcal{X}$ if and only if it satisfies the normalization condition:

$$\sum_{x \in \mathcal{X}} p(x) = 1$$

<u>Fundamental examples of discrete random variables</u>:

- Discrete Uniform Distribution: $X \sim \text{unif}(n)$

  ↳ Let $\mathcal{X} \subset \mathbb{R}$, $\mathcal{X} = \{x_1, \dots, x_n\}$

  The discrete uniform distribution:

  $$P(X = x_k) = \frac{1}{n}, \quad \forall\, k = 1, 2, \dots, n$$

Ex $X \sim \text{unif}(6)$ "rolling fair die once"

$\mathcal{X} = \{1, 2, 3, 4, 5, 6\}$ — range

$P(X = x_k) = \frac{1}{6} \quad \forall\, x_k \in \mathcal{X}$

- modes: all outcomes
- cdf: $F(x) = P(X \le x) = \min\left(\frac{\lfloor x \rfloor}{6}, 1\right)$

  for $x \ge 0$
- $\left.\begin{array}{l} P(X \le 3) = \frac{3}{6} = \frac{1}{2} \\ P(X < 3) = \frac{2}{6} = \frac{1}{3}, \quad \frac{1}{3} < \frac{1}{2} \end{array}\right\} \to$ 3 is a median

- <u>Bernoulli Random Variable</u>

Let (S,P) be a probability space, $E \subset S$ be an event with $P(E) = p$.

We refer to E as "success". Define $X: S \to \{0,1\}$ by:

$$X(s) = \begin{cases} 1, & \text{if } s \in E \\ 0, & \text{otherwise} \end{cases}$$

Then the pmf of X is

$$P(X=1) = p \quad \text{and} \quad P(X=0) = 1-p$$

Any discrete random variable X with the above pmf is called a Bernoulli random variable with success probability p and we will indicate this by $X \sim Ber(p)$

An experiment in which we observe if the given event E has occured is called a Bernoulli trial.

$\boxed{\text{Ex}}$ Biased coin lands heads with probability $p = 0.3$. Let $X = 1$ if the outcome is heads, and $X = 0$ otherwise.

$P(H) = 0.3$, $P(T) = 1 - 0.3 = 0.7$

$$X = \begin{cases} 1, & \text{if heads} \\ 0, & \text{if tails} \end{cases} \qquad X \sim Ber(0.3)$$

$P(X \leq 0) = 0.7$

$P(X \geq 1) = 0.3$

Mock: 0

Binomial Random Variable $X \sim Bin(n,p)$

- Consider a sequence of n independent $Ber(p)$ trials. Let X denote the number of successes observed during these n trials. Then:

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \forall\ k=0,1,\dots,n$$

Rmk: Note that $Ber(p) = Bin(1,p)$

- The binomial random variable can be mechanically simulated with the famous Galton board.

- Another way of viewing a binomial random variable

$$X_k \sim Ber(p) \text{ for } k=1,\dots,n$$

$$X = X_1 + X_2 + \dots + X_n \quad , \quad X \sim Bin(n,p)$$

- Thus any binomial random variable $X \sim Bin(n,p)$ is the sum of n independent Bernoulli random variables with the same probability of success p.

$\boxed{Ex}$ Fair die rolled 8 times. Let X be the number of times a 6 appears.

$$\hookrightarrow P(X=k) = \binom{8}{k}\left(\tfrac{1}{6}\right)^k\left(\tfrac{5}{6}\right)^{8-k} \quad k \in \{0,1,\dots 8\}$$

$$\text{so } P(X=0) = \binom{8}{0}\left(\tfrac{1}{6}\right)^0\left(\tfrac{5}{6}\right)^{8-0} = \left(\tfrac{5}{6}\right)^8$$

$$\text{so } P(X=2) = \binom{8}{2}\left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^6$$

$\boxed{\text{Ex}}$ Propogation of lies

- Line of 101 people
  - 100 of them say yes or no
  - Each person lies with probability $\frac{1}{2}$
  - What is probability last person heard the truth?

L = # of lies

P(P got the correct info") =

$P(L=0)+P(L=2)+P(L=4)+\ldots+P(L=100)$

Use Binomial random variable:

$L \sim \text{Bin}(100, \frac{1}{2})$

So pmf is: $P(X=K) = \binom{n}{K} p^K (1-p)^{n-K}$

$$\text{So } L = \binom{100}{0}\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^{100} + \binom{100}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{98} + \ldots + \binom{100}{100}\left(\tfrac{1}{2}\right)^{100}\left(\tfrac{1}{2}\right)^0$$

$$= \frac{1}{2^{100}}\left[\binom{100}{0}+\binom{100}{2}+\ldots+\binom{100}{100}\right]$$

$$\frac{1}{2^n}\left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots\right] = \frac{1}{2^n}\cdot 2^{n-1} = \frac{1}{2}$$

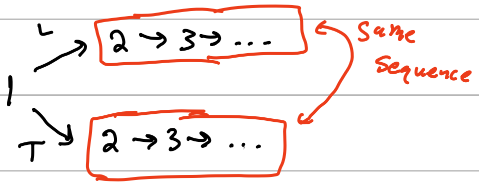
- So the probability that the last person hears the truth is $\frac{1}{2}$

- Also rememenber: $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots = 2^n$
- Also remember: $(1+1)^n = \binom{n}{0}+\binom{n}{1}+\ldots$
- Also remember: $0 = (1-1)^n = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots$

Summing these 2 you get 2^n

• Solving same problem in different way:

$P(L \text{ is even})$



- So the first decision decides whether the # of lies was even or odd.
- So the probability that the # of lies is even is $\frac{1}{2}$
- The diagram implies $P(L \text{ is even}) = P(L \text{ is odd}) = \frac{1}{2}$

• Geometric Distribution $X \sim Geom(p)$

- Repeat independent $Ber(p)$ trials until first success occurs. Let X be the number of trials required. Then $X \sim Geom(p)$. Its range is $\{1, 2, 3, \ldots\}$, and its pmf is:

$$P(X=k) = (1-p)^{k-1} p \quad , \quad k \geq 1 \quad (\text{all positive integers})$$

Prop: The geometric distribution satisfies the memoryless property:

$$P(X > n_0 + n \mid X > n) = P(X > n_0) \quad \forall \, n, n_0 \in \mathbb{Z}^+$$

$\boxed{\text{Ex}}$ Rolling a die until we get the first 6. Let X be the number of rolls required.

$$X \sim Geom\left(\tfrac{1}{6}\right)$$

pmf : $P(X=\frac{1}{6}) = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)$

$P(X=0) = 0$

$P(X=2) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{1} = \frac{5}{36}$

cdf : $F_X(k) = P(X \leq k) = P(X \leq k) = P(X=1) + P(X=2) + \ldots + P(X=k) = \sum_{n=1}^{k} p(1-p)^{n-1}$