Lecture 10

Reminder: The events A_1, A_2, A_3 are independent i.f.f.

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3) \quad \text{AND} \quad P(A_i \cap A_j) = P(A_i)P(A_j)$$

$$i \neq j, \quad i,j \in \{1,2,3\}$$

Rmk: One can prove that the random variable $X_1, \dots, X_n$ are independent i.f.f. for any sets $B_1, \dots, B_n \subseteq \mathbb{R}$ the events $\{X_1 \in B_1, \dots, X_n \in B_n\}$ are independent, i.e., $P(X_1 \in B_1, \dots, X_n \in B_n) = P(X_1 \in B_1) \cdot \ldots \cdot P(X_n \in B_n)$

$\boxed{\text{Ex}}$ Two numbers are chosen uniformly random in $[-1,2]$, and independent. What is the probability that their product is positive.

$X, Y \in [-1,2]$, $P(XY>0)$?

$$P(XY>0) = P(X \in [-1,0) \text{ AND } Y \in [-1,0)) + P(X \in (0,2] \text{ AND } Y \in (0,2])$$

$$= P(X \in [-1,0)) \cdot P(Y \in [-1,0)) + P(X \in (0,2]) \cdot P(Y \in (0,2])$$

$$= \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3}\left(\frac{2}{3}\right)$$

$$= \boxed{\frac{5}{9}}$$

Def : A random variable is discrete if its range is a finite or countable set of real numbers $x_1, x_2, \ldots$

$\boxed{\text{Ex}}$: flipping a fair coin twice, Let X be the number of heads.

$$X : \{HH, HT, TH, TT\} \to \underbrace{\{0,1,2\}}_{\text{Range of Random Variable}}$$

$X(HH) = 2$ $\quad P(X=2) = \frac{1}{4}$

$X(TT) = 0$ $\quad P(X=1) = \frac{2}{4} = \frac{1}{2}$

$X(HT) = 1$ $\quad P(X=0) = \frac{1}{4}$

Def : Let X be a discrete random variable with range $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. The probability mass Function (pmf) or the law of X is the function

$$p_X : X \to [0,1] \text{ given by } p_X(x_k) = \mathbb{P}(X = x_k)$$

- A mode of X is a value x^* in the range of X where $p_X(x)$ has a local maximum.

$\boxed{\text{Ex}}$ 2 fair Coins are tossed. Let X be # of heads obtained. Find probability mass function of X and determine its mode.

$p_X(0) = \frac{1}{4}$ $\quad$ The largest probability is $\frac{1}{2}$, and it occurs at 1.

$p_X(2) = \frac{1}{4}$

$p_X(1) = \frac{1}{2}$

Def: Two discrete random variables X and Y with ranges $\mathcal{X}$ and resp $\mathcal{Y}$ are called equivalent, and we denote this $X \sim Y$, if $\mathcal{X} = \mathcal{Y}$ and $p_X = p_Y$

$\boxed{\text{Ex}}$ Flip a fair coin.

$$X = \begin{cases} 1, & \text{if heads} \\ 0, & \text{if tails} \end{cases}$$

$P(X=1) = \frac{1}{2} = p_X(1)$

$P(X=0) = \frac{1}{2} = p_X(0)$

Roll a fair die

$$Y = \begin{cases} 1, & \text{if the outcome is odd} \\ 0, & \text{if the outcome is even} \end{cases}$$

$P(Y=1) = \frac{3}{6} = \frac{1}{2} = p_Y(1)$

$P(Y=0) = \frac{3}{6} = \frac{1}{2} = p_Y(0)$

- Range $\{0, 1\}$ for both
- $p_X(\cdot) = p_Y(\cdot)$
- So $X \sim Y$

$\boxed{\text{Ex}}$ A family has 3 children. Assume each child is independently a girl with probability $1/2$. Let G be the number of girls in the family.

a) Find range of G: $\{0, 1, 2, 3\}$

b) Compute probability mass function of G:

pmf: $p_G(0) = \frac{1}{8} = \frac{1}{2^3}$

$p_G(1) = \frac{3}{8}$

$p_G(2) = \frac{3}{8}$

$p_G(3) = \frac{1}{8}$

$p_G(x) = P(G = x)$

c) Find the median of G:

↳ median is $\frac{1}{2}$ - quantile

↳ $P(G \leq 0) = \frac{1}{8}$

↳ $P(G \leq 1) = \frac{3}{8}$

↳ $P(G \leq 0) + P(G \leq 1) = \frac{1}{2}$

↳ so 1 is the median

↳ "50% of families with 3 kids have at most one girl"

d) Find all modes of G:

↳ 1 and 2

$\boxed{\text{Ex}}$ A fair die rolled once. Outcome is <u>success</u> if a 6 appears. Define random variable X with range $\{0, 1\}$ by $X = 1$ if a success occurs, $X = 0$ otherwise.

Find probability mass function of X:

$$p_X(0) = P(X=0) = \frac{5}{6}$$

$$p_X(1) = P(X=1) = \frac{1}{6}$$

$\boxed{\text{Ex}}$ A fair die rolled n times. Let X be # of times a 6 appears. Find range of X and compute probability mass function!

- Range of X : $\{0, 1, \dots n\}$

- $p_X(k) = P(X=k) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{5}{6}\right)^{n-k}$, $k \in \{0, 1, \dots, n\}$

$\boxed{\text{Ex}}$ A fair die is rolled repeatedly until the 1st 6 appears. Let X be the number of rolls required.

a) Find Range of X : $\{1, 2, \dots, n, \dots\}$ (positive integers)

b) Compute probability mass function of X :

$p_X(k) = P(X=k) = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)$, k is the roll where the first 6 occurs

c) Compute the cumulative distribution function of X :

cdf: $F_X(x) = P(X \leq x)$

$= P(X=1) + \dots + P(X=n)$ $\forall n$ (integer, positive) and $x \in [n, n+1)$

$F(n)$: the probability that the first 6 appears after at most n rolls.

$P(X > n) = \left(\frac{5}{6}\right)^n$

$$\boxed{F(n) = 1 - \left(\frac{5}{6}\right)^n}$$