Lecture 1

Probability Spaces

- Some examples of chance experiments

  (i) flip a coin. The outcomes are Heads or Tails.

  (ii) Roll a dice. The possible outcomes are $S = \{1,2,3,4,5,6\}$

  (iii) Roll a pair of distinguishable dice. The possible outcomes are (n_1, n_2), $n_1, n_2 \in \{1,2,3,4,5,6\}$

  (iv) The amount of time you wait for the next bus. $t \in [0, \infty)$

  (v) Throw a dart. $S = \{(x,y) \in \mathbb{R}^2 \,;\, x^2+y^2 \leq r^2\}$

Def: The sample space of a random experiment/phenomenon is the set S of all possible outcomes.

Def: An event of a random experiment is a subset of the sample space.

$\boxed{\text{Ex}}$ Event = "the sum of observed numbers is 7 for 2 distinguishable die"

$$S_7 = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$S_7 \subset S$$

Def: The sample S itself is an event called the sure event.

Def: The empty set $\emptyset$ is called the impossible event.

- In concrete situations the events are described by properties:

  - the event of flipping a coin three times and obtaining at least two heads.

    $\{HHH, HHT, HTH, THH\}$

Set Operations: Recall, events are subsets!

Set Operations	linguistic counterparts
The union of $A \cup B$	A OR B (NOT EXLCUSIVE OR)
The intersection of $A \cap B$	A AND B
The complement A^c	NOT A
The set difference $A - B$	A BUT NOT B

$\boxed{\text{Ex}}$ Best of seven final

- OKC vs Pacers

For $K = 1, 2, \ldots, 7$, we define P_K to be the event "Pacers win game k"

- OKC wins game 1 $= P_1^c$
- Indiana wins one of the first two games $= P_1 \cup P_2$

- Indiana wins the series with at most 1 loss:

$$(P_1 \cap P_2 \cap P_3 \cap P_4) \cup (P_1^c, P_2, P_3, P_4, P_5) \cup (P_1, P_2^c, P_3, P_4, P_5) \cup (P_1, P_2, P_3^c, P_4, P_5)$$
$$\cup (P_1, P_2, P_3, P_4^c, P_5)$$

Def: Given a S - sample space. A probability function or probability distribution is a function

$$\mathbb{P} : S \to \mathbb{R}_{\{\geq 0\}}$$

$$\underset{\text{Event}}{E} \subset S \longmapsto \underbrace{\mathbb{P}(E)}_{\substack{\text{Probability}\\ \text{of}\\ \text{event } E}} \geq 0$$

Satisfying:

(i) $0 \leq \mathbb{P}(E) \leq 1 \quad \forall\ E \subset S$

(ii) $\mathbb{P}(\emptyset) = 0\ , \quad \mathbb{P}(S) = 1$

(iii) Countable additivity:

$$E_1, E_2, E_3, \ldots \qquad E_i \cap E_j \neq \emptyset \quad \left(\begin{smallmatrix}\text{for all}\\ i \neq j\end{smallmatrix}\right)$$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Def: A probability space is a pair $(\underset{\text{Sample space}}{S}, \underset{\text{probability function}}{\mathbb{P}})$

Def: An event E in a probability space $(S, \mathbb{P})$ is called almost sure (a.s.) if $\mathbb{P}(E) = 1$

Def: An event is called improbable if $\mathbb{P}(E) = 0$

$\boxed{\text{Ex}}$ Rolling a fair die.

$$S = \{1, 2, 3, 4, 5, 6\} \qquad E \subset S$$

$$\mathbb{P}(E) = \frac{|E|}{6}$$. So if E = "rolled even number" then: $\mathbb{P}(E) = \frac{1}{2}$

$\boxed{\text{Ex}}$ Uniform probability distribution S consists of N elements.

$$\forall\ s \in S \qquad \mathbb{P}_{unif}(\{s\}) = \frac{1}{N}$$

$$E \subset S \qquad \mathbb{P}_{unif}(E) = \frac{|E|}{N}$$

$\boxed{\text{Ex}}$ Weight Function

- if S is discrete (Finite or countable)

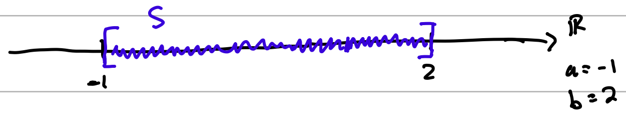
  $w: S \mapsto (0, \infty)$ on S, such that $Z_w = \sum_{s \in S} w(s) < \infty$

- We can define $\mathbb{P}_w$: $\mathbb{P}_w(\{s\}) = \frac{w(s)}{Z_w}$

- $\forall\ E \subset S$: $\mathbb{P}_w(E) = \frac{\sum_{s \in E} w(s)}{Z_w}$

- Note that $\mathbb{P}_w(E) = \sum_{s \in E} \mathbb{P}(s)$

$\boxed{\text{Ex}}$ Uniform distribution on interval. Let $S = [a, b]$



$A \subset S$

$$\mathbb{P}_{unif}(A) = \frac{\text{total length } (A)}{\text{total length } (S)} = \frac{\text{total length } (A)}{(b-a)}$$

- Probability of negative number in $[-1, 2]$ = $\frac{\text{length of } (A)}{\text{length of } (S)} = \frac{1}{3}$

- Probability that the number is $\frac{1}{2}$ in $[-1, 2]$ = $\frac{\text{length of } (A)}{\text{length of } (S)} = \frac{0}{3} = 0$ because $A = [\frac{1}{2}, \frac{1}{2}]$

  ↳ so this event is improbable but not impossible by def.

$\boxed{\text{Ex}}$ if S is a region in the plane

$$\mathbb{P}_{unif}(A) = \frac{\text{Area}(A)}{\text{Area}(S)}$$